MIGSAA Extended Project

Fourier Restriction Norm Method

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Abstract

In 1993, Bourgain introduced the Fourier Restriction Spaces, or $X^{s,b}$, in order to prove local well-posedness of nonlinear Schrödinger and Korteweg-de Vries type equations, through a contraction mapping argument. By taking advantage of the localisation of the space-time Fourier Transform of the linear solution, and certain properties of the nonlinearity, it led to improvements on previous results. Through the years, the idea was followed by many authors and further developed, leading to the Fourier Restriction Norm Method.

We present a selection of known results, for cubic and quintic nonlinear Schrödinger equation, in one dimension, and for Korteweg-de Vries type equation, in the periodic setting. In particular, we focus on linear estimates for general dispersive equations, and proceed to equation specific estimates, such as Strichartz and nonlinear estimates.

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Contents

1 Introduction

In 1993, Jean Bourgain proposed a harmonic analytic method to prove local well-posedness (LWP) of nonlinear evolution equations, in the periodic setting. The focus was on Nonlinear Schrödinger (NLS) and Korteweg-de Vries type equations, using a contraction mapping argument in spaces that exploit both linear and nonlinear properties of the equation. The method was picked up by many authors and further developed, known as the *Fourier Restriction Norm Method*.

At the heart of the method is the nonlinear solution as a perturbation of the linear one. The underlying spaces, *Fourier restriction norm spaces* or $X^{s,b}$, capture the localisation of the space-time Fourier transform of the linear solution, and the fact that certain nonlinearities have a weaker effect than the dispersion on the solution, at least for a short time.

It is relevant to point out that the modern version of $X^{s,b}$-spaces is due to Bourgain, but it was first mentioned in the works of Rauch-Reed [7] (one dimension) and Beals [1] (higher dimensions) in the context of the wave equation. However, they were only used in the context of LWP in 1993, by Bourgain [2], for NLS and KdV, and Klainerman-Machedon [8], for the wave equation.

Our goal is to study the Fourier Restriction Norm Method applied to NLS and KdV, in the periodic setting, closely following results in [2, 4, 5, 6, 9]. We start by introducing the relevant equations. The nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1} u \\ u|_{t=0} = u_0 \end{cases} \quad , \quad u : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C} \tag{NLS}$$

is of great importance in Physics, used to describe propagation of light in nonlinear optical fibers, Bose-Einstein condensates among many others. Similarly, the Korteweg-de Vries type equations

$$\begin{cases} \partial_t u + \partial_x^3 u = \pm u^p \partial_x u \\ u|_{t=0} = u_0 \end{cases} \quad , \quad u : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \tag{KdV}$$

describe, for instance, the propagation of long waves in shallow water and plasma physics. They are classified depending on the nonlinearity: $p = 1$, Korteweg-de Vries equation (KdV); $p = 2$, modified Korteweg-de Vries equation (mKdV); $p \geq 3$, generalised Korteweg-de Vries equation (gKdV).

Regarding NLS, the LWP results in [2] can be summarised as follows: NLS is LWP in $H^s(\mathbb{T}^d)$ with

- $d = 1$, $p - 1 < \frac{4}{1-2s}$;
- $d = 2$, $p - 1 \geq 2$ and sufficiently small data;
- $d = 3$, $p = 3$, $s > \frac{1}{2}$;
- $d \geq 4$, $d \geq 4$ for $2 \leq p - 1 < \frac{4}{d-2s}$ and $s > \frac{3d}{d+4}$.

Similarly, for KdV, Fourier Restriction Norm Method was used for the following results: the following equations are LWP in $H^s(\mathbb{T})$

- KdV for $s > 0$ [2] and $s > -\frac{1}{2}$ [4, 6];
- mKdV for $s > \frac{1}{2}$ [2];
- gKdV for $s > \frac{1}{2}$ [5].

For a space $\mathcal{X} \subset C_t H^s_x$, we intend to show the existence of a finite time T such that for all $u_0 \in H^s(\mathbb{T}^d)$,

1. Existence: there exists u in $\mathcal{X}([0,T] \times \mathbb{T}^d)$ which satisfies the equation, with initial data u_0,
2. Uniqueness: u is a unique solution in $\mathcal{X}([0,T] \times \mathbb{T}^d)$,
3. Continuous dependence on initial data: the map $\Gamma : H^s(\mathbb{T}^d) \to \mathcal{X}([0,T] \times \mathbb{T}^d)$ from initial data to solutions, is locally uniformly continuous with respect to initial data.

When considering a contraction mapping argument, showing LWP reduces to proving that the solution map is a contraction in an appropriate space and proving uniform continuity of the map. To that end, one requires an adequate space for the linear solution, $X^{s,b}$ and linear estimates, a space for the nonlinearity and nonlinear estimates.

In Chapter 2, we introduce the framework of the method, focusing on the linear equation and the contraction mapping argument. The linear equation is treated in generality, for any constant coefficients differential operator, with introduction of the Duhamel's formulation. The localisation of the space-time Fourier transform of the linear solution motivates the introduction of the $X^{s,b}$-spaces. The relevant properties of such spaces (duality, embeddings, behaviour under translations and complex conjugation) are mentioned to improve their understanding. In addition, the homogeneous and inhomogeneous linear estimates are proven, as well as auxiliary results (Time Localisation estimate and Transference Principle), reducing the problem of LWP to showing nonlinear estimates.

Chapter 3 focuses on the nonlinear Schrödinger equation, namely cubic and quintic on the one-dimensional torus. We present Strichartz estimates introduced by Bourgain, needed for the nonlinear estimates. Lastly, the LWP results are stated and proven.

In Chapter 4, we introduce the analogous theory for KdV. It requires similar spaces $Y^{s,b}$, needed for the contraction mapping argument. The nonlinearity is treated by two estimates: the bilinear estimate (by Kenig, Ponce and Vega) and a multilinear estimate, needed for mKdV and gKdV (due to Colliander, Keel, Staffilani, Takaoka and Tao). The bilinear estimate imposes certain conditions on the nonlinearity, which are shown to be necessary through counter-examples.

2 Framework

The Fourier Restriction Norm Method uses a contraction mapping argument to show local well-posedness of nonlinear evolution equations. We reduce solving the Partial Differential Equation (PDE) to showing that the solution map Γ_{u_0} has a unique fixed point u, $\Gamma_{u_0}(u) = u$. Thus, it suffices to fulfil the conditions for the Banach Fixed Point Theorem, i.e., Γ_{u_0} is a contraction in an appropriate space. Moreover, we require Γ_{u_0} to have continuous dependence on the initial data u_0.

Let $B_R \subset X^{s,b}$ a ball of radius R. It suffices to show that

- $\Gamma_{u_0} : B_R \to B_R$,
- Γ_{u_0} is Lipschitz continuous in B_R, with constant less than 1,
- Γ_{u_0} is uniformly continuous with respect to the initial data u_0.

In this chapter, we introduce the framework for the method, in the periodic setting, for a general evolution equation, with linear differential operator with constant coefficients. Firstly, we introduce the $X^{s,b}$ spaces and the linear estimates, homogeneous and inhomogeneous. Secondly, we prove additional estimates needed to complete the argument: Time Localisation Estimate (to improve control over the nonlinear part) and Transference Principle (for nonlinear estimates). Lastly, we clarify the application to the nonlinear equation and reduce the problem to determining nonlinear estimates.

The interested reader can find additional details in [9].

2.1 Preliminaries

We start by recalling the definition of the Fourier Transform and of some important functional spaces. The spatial Fourier Transform and the associated space-time Fourier Transform are one of the main tools of the method. We are concerned with periodic functions in the spatial variable, namely with period 2π defined on the d-dimensional torus $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$.

We start by defining the *Schwartz space*, $\mathcal{S}_x(\mathbb{T}^d)$, as the space of smooth functions $f : \mathbb{T}^d \to \mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, such that

$$\left\| x^\alpha \partial_x^\beta f(x) \right\|_{L_x^\infty(\mathbb{T}^d)} < \infty, \ \forall \alpha \in \mathbb{Z}_{>0}, \beta \in \mathbb{Z}_{\geq 0}^d.$$

The *spatial Fourier Transform* is defined as an operator $\mathcal{F} : \mathcal{S}_x\left(\mathbb{T}^d\right) \to \mathcal{S}_n\left(\mathbb{Z}^d\right)$, defined as follows: for $f \in \mathcal{S}_x\left(\mathbb{T}^d\right)$, its Fourier Transform is given by

$$\mathcal{F}(f)(n) = \hat{f}(n) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} \, dx, \ n \in \mathbb{Z}^d.$$

The Fourier Transform is an invertible operator on Schwartz functions. Thus, in the periodic setting, we can define an *inversion formula* given by a Fourier series

$$f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in\cdot x}.$$

The Fourier Transform can be extended to an operator between $L^2_x\left(\mathbb{T}^d\right)$ and $\ell^2_n\left(\mathbb{Z}^d\right)$, in the same way.

Similarly, we define the Schwartz space $\mathcal{S}_{t,x}\left(\mathbb{R}\times\mathbb{T}^d\right)$ and the *space-time Fourier Transform* and respective inversion formula

$$\mathcal{F}(u)(\tau,n)=\widehat{u}(\tau,n):=\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}}\int_{\mathbb{T}^d}u(t,x)e^{-i(t\tau+x\cdot n)}\,dx\,d\tau,$$
$$u(t,x)=\sum_{n\in\mathbb{Z}^d}\int_{\mathbb{R}}\hat{u}(\tau,n)e^{i(t\tau+x\cdot n)}\,d\tau,$$

for $\tau\in\mathbb{R},n\in\mathbb{Z}^d$. In addition, this operator can also be extended to an operator $\mathcal{F}:L^2_{t,x}\left(\mathbb{R}\times\mathbb{T}^d\right)\to\ell^2_n\left(\mathbb{Z}^d\right)\times L^2_\tau\left(\mathbb{R}\right)$, in the same fashion.

Recall the definition of *Sobolev spaces* $W^{s,p}_x\left(\mathbb{T}^d\right)$, $W^{r,q}_t\left(\mathbb{R}\right)$, $s,r\in\mathbb{R}$, $1\le p,q<\infty$ as the completion of $\mathcal{S}_x\left(\mathbb{T}^d\right)$ and $\mathcal{S}_t\left(\mathbb{R}\right)$, with respect to the following norms, respectively,

$$\|f\|_{W^{s,p}_x\left(\mathbb{T}^d\right)}:=\left\|\langle n\rangle^s\widehat{f}(n)\right\|_{\ell^2_n\left(\mathbb{Z}^d\right)},$$
$$\|g\|_{W^{r,q}_t\left(\mathbb{R}\right)}:=\|\langle\tau\rangle^r\widehat{g}(\tau)\|_{L^2_\tau\left(\mathbb{R}\right)},$$

with $\langle\cdot\rangle=\left(1+|\cdot|^2\right)^{\frac{1}{2}}$ denoting *japanese brackets*. Note that if $p,q=2$, we use the following notation $W^{s,2}_x\left(\mathbb{T}^d\right)=H^s_x\left(\mathbb{T}^d\right)$ and $W^{r,2}_t\left(\mathbb{R}\right)=H^r_t\left(\mathbb{R}\right)$.

From now on, we will omit the underlying space when referring to functional spaces, whenever it is clear from the context, but will index the space by the corresponding variable.

Lastly, we introduce the following notation,

- $A\lesssim B$ if there exists $C>0$ such that $A\le CB$;
- $A\sim B$ if $A\lesssim B$ and $A\gtrsim B$;
- $A\ll B$ if there exists $0<C<1$ such that $A\le CB$.

When relevant, $\lesssim,\sim$ are indexed to represent the dependence of the constant C on a certain quantity.

2.2 Linear Equation

The Fourier Restriction Norm Method uses a contraction mapping argument, thus considering the nonlinear solution as a perturbation of the linear one. Such a strategy is reasonable for short times, as certain nonlinearities do not significantly alter the solution for short times, after time localisation. Therefore, we start by focusing on a family of linear evolution equations.

Consider the following equation, defined on the d-dimensional torus,

$$\begin{cases}\partial_t u(t,x)=\mathcal{L}u(t,x)+F(t,x)\\ u(0,x)=u_0(x)\end{cases},\quad u:\mathbb{R}\times\mathbb{T}^d\to\mathbb{K},\tag{2.1}$$

with $\mathcal{L}$ a differential operator, F a smooth function and $\mathbb{K}$ either $\mathbb{C}$ or $\mathbb{R}$, depending on the equation.

We are interested in differential operators $\mathcal{L}$ with constant coefficients, with the form $\mathcal{L} = ih\left(\nabla/i\right)$, for some polynomial h, i.e.,

$$\mathcal{L}u := i \sum_{|\alpha|\leq k} i^{|\alpha|-1} c_\alpha \partial_x^\alpha u,$$

with $k \in \mathbb{Z}_{>0}$ the order of the operator, $c_\alpha \in \mathbb{C}$ for each $\alpha \in \mathbb{Z}^d_{\geq 0}$ with $|\alpha| \leq k$.

Remark. For NLS and KdV type equations, we have

$$\begin{aligned} \text{NLS}: \quad & \mathcal{L} = i \sum_{j=1}^{d} \partial_{x_j}^2 \quad \text{on } \mathbb{T}^d, \\ \text{KdV}: \quad & \mathcal{L} = -\partial_x^3 \quad \text{on } \mathbb{T}. \end{aligned}$$

Considering the representation $\mathcal{L} = ih(\nabla/i)$, h is called the *dispersion relation* of $\mathcal{L}$ and has the following form for the equations at hand,

$$\begin{aligned} \text{NLS}: \quad & h(\xi) = -|\xi|^2, \\ \text{KdV}: \quad & h(\xi) = \xi^3. \end{aligned}$$

Note that the $\mathcal{L} = ih(\nabla/i)$ representation of the differential operator is useful when applying Fourier Transform,

$$\begin{aligned} \mathcal{F}\{\mathcal{L}u\}(n) &= \mathcal{F}\{ih(\nabla/i)u\}(n) \\ &= \mathcal{F}\left\{ i \sum_{|\alpha|\leq k} i^{|\alpha|-1} c_\alpha \left(\nabla/i\right)^\alpha u \right\}(n) \\ &= \mathcal{F}\left\{ \sum_{|\alpha|\leq k} c_\alpha \nabla^\alpha u \right\}(n) \\ &= \sum_{|\alpha|\leq k} c_\alpha i^{|\alpha|} n^\alpha \widehat{u}(n) \\ &= ih(n)\widehat{u}(n). \end{aligned}$$

A standard technique to solve Partial Differential Equations (PDEs) is applying a spatial Fourier Transform in order to reduce it to an Ordinary Differential Equation (ODE) in time. Using the previous result on the Fourier Transform of the differential operator $\mathcal{L}$, we can reduce (2.1) to

$$\mathcal{F}\left\{\partial_t u\right\}(t,n) = \mathcal{F}\left\{\mathcal{L}u + F\right\}(t,n) \iff \partial_t \widehat{u}(t,n) = ih(n)\widehat{u}(t,n) + \widehat{F}(t,n).$$

Thus

$$\begin{aligned} \widehat{u}(t,n) &= e^{ith(n)}\widehat{u_0}(n) + \int_0^t e^{i(t-t')h(n)} \widehat{F}(t',n)\, dt', \\ u(t,x) &= e^{t\mathcal{L}} u_0(x) + \int_0^t e^{(t-t')\mathcal{L}} F(t',x)\, dt', \end{aligned} \tag{2.2}$$

with the operator $e^{t\mathcal{L}}$ defined as follows

$$\left(e^{t\mathcal{L}} f\right)(t,x) = \sum_{n\in\mathbb{Z}^d} \left(e^{ith(n)} \widehat{f}(n)\right) e^{in\cdot x}.$$

The expression in (2.2) is called *Duhamel's formula* and the operator $e^{t\mathcal{L}}$ is called *linear propagator*, often denoted as $S(t)$.

Moreover, if we consider $F \equiv 0$ in (2.1) and apply a space-time Fourier Transform,

$$\begin{aligned} 0 &= \mathcal{F}\left\{\partial_t u - \mathcal{L}u\right\}(\tau, n) \\ &= i\left(\tau - h(n)\right)\widehat{u}(\tau, n), \end{aligned}$$

it follows that the space-time Fourier Transform of u is a measure supported on the characteristic hypersurface $\left\{(\tau, n) \in \mathbb{R} \times \mathbb{Z}^d : \ \tau = h(n)\right\}$,

$$\widehat{u}(\tau, n) = \delta(\tau - h(m))\widehat{u_0}(n),$$

with δ the Dirac delta function.

Recalling the perturbation approach, we expect the space-time Fourier Transform of a Picard iterate to have support close to this hypersurface, at least for a short time, when suitably time-localised.

2.3 $X^{s,b}$ Spaces

In this section, we focus on the $X^{s,b}$ spaces and some of their relevant properties.

Definition 2.1. Let $h : \mathbb{T}^d \to \mathbb{R}$ a continuous function, and let $s, b \in \mathbb{R}$. The space $X^{s,b}_{\tau=h(n)}\left(\mathbb{R} \times \mathbb{T}^d\right)$, abbreviated $X^{s,b}$, is defined as the closure of the Schwartz functions $\mathcal{S}_{t,x}\left(\mathbb{R} \times \mathbb{T}^d\right)$ under the *norm*

$$\|u\|_{X^{s,b}} := \left\|\langle n\rangle^s \langle \tau - h(n)\rangle^b \widehat{u}(\tau, n)\right\|_{\ell^2_n L^2_\tau} = \left\|e^{-t\mathcal{L}}u\right\|_{H^b_t H^s_x},$$

with '$e^{-t\mathcal{L}}u$' the *interaction representation.*

Note that the underlying hypersurface to $X^{s,b}$ will be omitted if it is clear from the context.

The $X^{s,b}$-norm captures the expected localisation of the perturbed solution through $\langle \tau - h(n)\rangle^b$, as it penalises the function if it lies far from the hypersurface.

The norm can be seen as a weighted Sobolev norm, sharing equivalent properties.

Proposition 2.2.

1. *$X^{s',b'} \subset X^{s,b}$ for $s' \geq s$, $b' \geq b$.*
2. $\left(X^{s,b}_{\tau=h(n)}\right)^* = X^{-s,-b}_{\tau=-h(-n)}$.
3. *$X^{s,b}$ is invariant under translations in space and time.*
4. $\|\bar{u}\|_{X^{s,b}_{\tau=-h(-n)}} = \|u\|_{X^{s,b}_{\tau=h(n)}}$.

The $X^{s,b}$ spaces are a useful to conduct a contraction mapping argument since they are a subset of $C_t H^s_x$, under certain conditions, as stated in the following proposition.

Proposition 2.3. *The following embedding holds for all $s \in \mathbb{R}$ and $b > \frac{1}{2}$,*

$$X^{s,b} \subset C_t H^s_x.$$

Proof. Let $u \in X^{s,b}$. We start by proving that $\|u\|_{L_t^\infty H_x^s} < \infty$. Note that $e^{-t\mathcal{L}}$ is unitary in H_x^s, for any $f \in H_x^s$

$$\|e^{-t\mathcal{L}} f\|_{H_x^s} = \|\langle n\rangle^s e^{-ith(n)} \widehat{f}(n)\|_{\ell_n^2} = \|f\|_{H_x^s}.$$

Thus, using the interaction representation for the $X^{s,b}$-norm and Sobolev embedding in time, for any $b > \frac{1}{2}$

$$\|u\|_{L_t^\infty H_x^s} = \|e^{i\mathcal{L}} u\|_{L_t^\infty H_x^s} \lesssim \|e^{t\mathcal{L}} u\|_{H_t^b H_x^s} = \|u\|_{X^{s,b}}.$$

It remains to prove that u is continuous in time. Let $t \in \mathbb{R}$ and $\{t_k\} \subset \mathbb{R}$ a sequence which converges to t as $k \to \infty$. Then, using inverse Fourier Transform in time and Cauchy-Schwarz inequality

$$\begin{aligned}
\|u(t_k) - u(t)\|_{H_x^s} &= \|\langle n\rangle^s \left(\widehat{u}(t_k, n) - \widehat{u}(t, n)\right)\|_{\ell_n^2} \\
&= \left\| \int_{\mathbb{R}} \langle n\rangle^s \widehat{u}(\tau, n) \left(e^{it_k\tau} - e^{it\tau}\right) \, d\tau \right\|_{\ell_n^2} \\
&= \left\| \int_{\mathbb{R}} \langle n\rangle^s \frac{\langle \tau - h(n)\rangle^b}{\langle \tau - h(n)\rangle^b} \widehat{u}(\tau, n) \left(e^{it_k\tau} - e^{it\tau}\right) \, d\tau \right\|_{\ell_n^2} \\
&\lesssim \left\| \left(\int_{\mathbb{R}} \langle \tau - h(n)\rangle^{-2b} \, d\tau \right)^{\frac{1}{2}} \langle n\rangle^s \langle \tau - h(n)\rangle^b \widehat{u}(\tau, n) \left(e^{it_k\tau} - e^{it\tau}\right) \right\|_{\ell_n^2 L_\tau^2} \\
&\lesssim \|u\|_{X^{s,b}} < \infty,
\end{aligned}$$

which follows from the τ-integral being finite ($-2b < -1$) and $\left|e^{it_k\tau} - e^{it\tau}\right| \leq 2$. Note that the implicit constants do not depend on k, t or t_k. Therefore, we can apply the dominated convergence theorem, using the fact that $\left|e^{it_k\tau} - e^{it\tau}\right| \to 0$, proving that

$$\|u(t_k) - u(t)\|_{H_x^s}^2 \to 0,$$

as $k \to \infty$, as intended. □

2.4 Linear Estimates

To conduct a contraction mapping argument in $X^{s,b}$, we require linear estimates to control the homogeneous linear solution and the integral part in the Duhamel's formula. In this section we state and prove the homogeneous and inhomogeneous linear estimates.

$X^{s,b}$ will be the underlying space for the contraction mapping argument, therefore the linear solution $e^{t\mathcal{L}} u_0$, with $u_0 \in H_x^s$, must live in $X^{s,b}$. Unfortunately, this is not the case

$$\left\|e^{t\mathcal{L}} u_0\right\|_{X^{s,b}} = \left\|e^{-t'\mathcal{L}} e^{t\mathcal{L}} u_0\right\|_{H_{t'}^b H_x^s} = \left\|\|e^{(t-t')\mathcal{L}} u_0\|_{H_x^s}\right\|_{H_{t'}^b} = \left\|\|u_0\|_{H_x^s}\right\|_{H_{t'}^b} = \infty,$$

using that $e^{t\mathcal{L}}$ is an isometry in H_x^s, $\|u_0\|_{H_x^s} < \infty$ but the time norm is considered over the whole space $\mathbb{R}$.

As a consequence, we need to localise our solution in time, so that it belongs to $X^{s,b}$. To that end we introduce a smooth cutoff function, $\eta \in \mathcal{S}_t$, with the following properties

- $\text{supp}(\eta) \subset (-2, 2)$,
- $\eta(t) = 1$ for $t \in [-1, 1]$,

- $\eta(t) = \eta(-t)$ and $\eta(t) \geq 0$ for all $t \in \mathbb{R}$.

To account for the time localisation, we modify the solution map accordingly,

$$\Gamma_{u_0}(u) := \eta(t)e^{t\mathcal{L}}u_0 + \eta\left(\frac{t}{T}\right)\int_0^t e^{(t-t')\mathcal{L}}F(t')\,dt',$$

for some $0 < T < 1$. The need to apply a time cutoff raises the following question: if $u \in X^{s,b}$, does ηu also lie in $X^{s,b}$? In fact, $X^{s,b}$ is stable with respect to time localisation, as indicated by the following lemma.

Lemma 2.4 (Stability of $X^{s,b}$ with respect to time localisation)**.** *Let $\eta \in \mathcal{S}_t$ a smooth cutoff and $u \in X^{s,b}$. Then,*

$$\|\eta(t)u\|_{X^{s,b}} \lesssim_{\eta,b} \|u\|_{X^{s,b}}, \quad \forall s, b \in \mathbb{R}. \tag{2.3}$$

Proof. Using Fourier inversion formula on η and Minkowski's integral inequality

$$\begin{aligned}
\|\eta(t)u\|_{X^{s,b}} &= \left\|\left(\int_{\mathbb{R}} \widehat{\eta}(\tau_0)e^{it\tau_0}\,d\tau_0\right)u\right\|_{X^{s,b}} \\
&\lesssim \int_{\mathbb{R}} \left\|\widehat{\eta}(\tau_0)e^{it\tau_0}u\right\|_{X^{s,b}}\,d\tau_0 \\
&= \int_{\mathbb{R}} |\widehat{\eta}(\tau_0)|\left\|e^{it\tau_0}u\right\|_{X^{s,b}}\,d\tau_0 \\
&= \int_{\mathbb{R}} |\widehat{\eta}(\tau_0)|\left\|\langle n\rangle^s\langle\tau - h(n)\rangle^b\widehat{u}(\tau-\tau_0,n)\right\|_{\ell^2_n L^2_\tau}\,d\tau_0.
\end{aligned}$$

Applying the change of variables $\tau' = \tau - \tau_0$ and $\langle A+B\rangle^b \lesssim_b \langle A\rangle^{|b|}\langle B\rangle^b$, gives

$$\begin{aligned}
\|\eta(t)u\|_{Xs,b} &= \int_{\mathbb{R}} |\widehat{\eta}(\tau_0)|\left\|\langle n\rangle^s\langle\tau' + \tau_0 - h(n)\rangle^b\widehat{u}(\tau',n)\right\|_{\ell^2_n L^2_{\tau'}}\,d\tau_0 \\
&\lesssim_b \int_{\mathbb{R}} |\widehat{\eta}(\tau_0)|\,\langle\tau_0\rangle^{|b|}\left\|\langle n\rangle^s\langle\tau' - h(n)\rangle^b\widehat{u}(\tau',n)\right\|_{\ell^2_n L^2_{\tau'}}\,d\tau_0 \\
&\lesssim_\eta \|u\|_{X^{s,b}},
\end{aligned}$$

since $\eta \in \mathcal{S}_t$, concluding the proof. □

Modifying the solution map to be localised in time does not influence the estimates in $X^{s,b}$, thus we can focus on showing that Γ_{u_0} is a contraction in $X^{s,b}$. We aim to show

$$\begin{aligned}
\left\|\eta(t)e^{t\mathcal{L}}u_0\right\|_{X^{s,b}} &\lesssim \|u_0\|_{H^s_x}, \\
\|\mathcal{I}(F)\|_{X^{s,b}} &\lesssim \|F\|_Y,
\end{aligned}$$

with $\mathcal{I}(F) := \int_0^t e^{(t-t')\mathcal{L}}F(t')\,dt'$ and Y an appropriate space. These are called *linear estimates*, *homogeneous* and *inhomogeneous*, respectively.

2.4.1 Homogeneous Linear Estimate

The following estimate controls the time-localised linear solution with the norm of the initial data.

Proposition 2.5. *Let $u_0 \in H^s_x$. Then,*

$$\|\eta(t)e^{t\mathcal{L}}u_0\|_{X^{s,b}} \lesssim_{\eta,b} C\|u_0\|_{H^s_x}. \tag{2.4}$$

Proof. We start by calculating the space-time Fourier transform of $\eta e^{t\mathcal{L}} u_0$,

$$\begin{aligned}\mathcal{F}\left(\eta(t)e^{t\mathcal{L}}u_0\right)(\tau,n) &= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}}\int_{\mathbb{T}^d} e^{-i(t\tau+x\cdot n)}\eta(t)e^{ith(n)}u_0(x)\,dx\,dt \\ &= \left\{\frac{1}{2\pi}\int_{\mathbb{R}} e^{-it(\tau-h(n))}\eta(t)\,dt\right\}\left\{\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-ix\cdot n}u_0(x)\,dx\right\} \\ &= \widehat{\eta}(\tau-h(n))\widehat{u_0}(n).\end{aligned}$$

Now we can calculate the norm,

$$\begin{aligned}\|\eta(t)e^{t\mathcal{L}}u_0\|_{X^{s,b}} &= \left\|\langle n\rangle^s\langle\tau-h(n)\rangle^b\widehat{\eta}(\tau-h(n))\widehat{u_0}(n)\right\|_{\ell^2_n L^2_\tau} \\ &= \left\|\left\|\langle\tau-h(n)\rangle^b\widehat{\eta}(\tau-h(n))\right\|_{L^2_\tau}\langle n\rangle^s\widehat{u_0}(n)\right\|_{\ell^2_n} \\ &= \|\eta\|_{H^b_t}\,\|\langle n\rangle^s\widehat{u_0}\|_{\ell^2_n} \\ &= \|\eta\|_{H^b_t}\,\|u_0\|_{H^s_x} \\ &\lesssim_{\eta,b} \|u_0\|_{H^s_x},\end{aligned}$$

since $\eta\in\mathcal{S}_t$. □

2.4.2 Inhomogeneous Linear Estimate

Recall that $\mathcal{I}(F)=\int_0^t e^{(t-t')\mathcal{L}}F(t')\,dt'$ denotes the integral part of the linear solution in Duhamel's formula. We need to guarantee that $\mathcal{I}(F)$ also lives in some $X^{s,b}$ space, as stated in Proposition 2.6.

Proposition 2.6. *Let $s\in\mathbb{R}$, $b>\frac{1}{2}$, $F\in X^{s,b-1}$ and $\eta\in\mathcal{S}_t$ a smooth cutoff function. Then,*

$$\|\eta(t)\mathcal{I}(F)\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}}. \tag{2.5}$$

Proof. We start by calculating the space-time Fourier transform of $\eta(t)\mathcal{I}(F)$. Ignoring the factors of 2π, applying a Fourier transform in space gives

$$\begin{aligned}\mathcal{F}\left\{\eta(t)\mathcal{I}(F)\right\}(t,n) &= \mathcal{F}\left\{\eta(t)\int_0^t e^{(t-t')\mathcal{L}}F(t',x)\,dt'\right\}(t,n) \\ &= \eta(t)\int_0^t e^{i(t-t')h(n)}\widehat{F}(t',n)\,dt',\end{aligned}$$

and lastly in time,

$$\begin{aligned}\mathcal{F}\left\{\eta(t)\mathcal{I}(F)\right\}(\tau,n) &= \int_{\mathbb{R}}\eta(t)e^{-it\tau}\int_0^t e^{i(t-t')h(n)}\widehat{F}(t',n)\,dt'\,dt \\ &= \int_{\mathbb{R}}\eta(t)e^{-it(\tau-h(n))}\int_{\mathbb{R}}\mathbb{1}_{[0,t]}(t')e^{-it'h(n)}\widehat{F}(t',n)\,dt'\,dt \\ &= \int_{\mathbb{R}}\eta(t)e^{-it(\tau-h(n))}\mathcal{F}_{t'\to\tau'}\left(\mathbb{1}_{[0,t]}(t')\widehat{F}(t',n)\right)_{\tau'=h(n)}\,dt'\,dt \\ &= \int_{\mathbb{R}}\eta(t)e^{-it(\tau-h(n))}\left(\widehat{\mathbb{1}}_{[0,t]}(t')\star_{\tau'}\widehat{F}(\cdot,n)\right)_{\tau'=h(n)}\,dt,\end{aligned}$$

with $\star_{\tau'}$ representing convolution on variable τ'. Since

$$\widehat{\mathbb{1}}_{[0,t]}(t') = \int_0^t e^{-i\tau t'}\,d\tau = -\frac{1}{i\tau}e^{-i\tau t'}\Big]_0^t = \frac{e^{-itt'}-1}{it},$$

it follows that

$$\begin{aligned}\mathcal{F}\left\{\eta(t)\mathcal{I}(F)\right\}(t,n) &= \int_{\mathbb{R}} \eta(t) e^{-it(\tau-h(n))} \int_{\mathbb{R}} \frac{e^{-it(h(n)-\tau')}-1}{-i(h(n)-\tau')} \widehat{F}(\tau',n)\, d\tau'\, dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(e^{-it(\tau-h(n))} - e^{-it(\tau-\tau')}\right) \eta(t)\, dt\, \frac{1}{-i(\tau'-h(n))} \widehat{F}(\tau',n)\, d\tau' \\ &= (2\pi) \int_{\mathbb{R}} \left(\widehat{\eta}(\tau-h(n)) - \widehat{\eta}(\tau-\tau')\right) \frac{1}{-i(\tau'-h(n))} \widehat{F}(\tau',n)\, d\tau'.\end{aligned}$$

Without loss of generality, we can assume $s=0$, otherwise we can run the argument for $\widehat{G}(n) = \langle n\rangle^s \widehat{F}(n)$. We will consider two frequency regions for the Fourier Transform, R_1 where $|\tau'-h(n)| \leq 1$ and R_2 where $|\tau'-h(n)| \geq 1$.

<u>Case 1:</u> $|\tau'-h(n)| \leq 1$

Calculating the $X^{0,b}$ norm only considering frequencies in R_1, note that $\langle \tau-h(n)\rangle \leq \langle \tau-\tau'+\alpha\rangle + \langle \tau'-h(n)-\alpha\rangle \lesssim \langle \tau-\tau'+\alpha\rangle$, for α between 0 and $\tau'-h(n)$,

$$\begin{aligned} I &:= \left\| \langle \tau-h(n)\rangle^b \int_{\mathbb{R}} \left(\widehat{\eta}(\tau-h(n)) - \widehat{\eta}(\tau-\tau')\right) \frac{1}{-i(\tau'-h(n))} \widehat{F}(\tau',n)\, d\tau' \right\|_{\ell^2_n L^2_\tau(R_1)} \\ &= \left\| \int_{\mathbb{R}} \int_0^{\tau'-h(n)} \langle \tau-h(n)\rangle^b \widehat{\eta}'(\tau-\tau'+\alpha) \frac{\widehat{F}(\tau',n)}{\tau'-h(n)}\, d\alpha\, d\tau' \right\|_{\ell^2_n L^2_\tau(R_1)} \\ &\lesssim_b \left\| \int_{\mathbb{R}} \int_0^{\tau'-h(n)} \langle \tau-\tau'+\alpha\rangle^b \widehat{\eta}'(\tau-\tau'+\alpha) \frac{\widehat{F}(\tau',n)}{\tau'-h(n)}\, d\alpha\, d\tau' \right\|_{\ell^2_n L^2_\tau(R_1)}. \end{aligned}$$

Using Minkowski's integral inequality on the τ-integral and the sum, gives

$$\begin{aligned} I &\lesssim_b \left\| \int_{R_1} \int_0^{\tau'-h(n)} \frac{\widehat{F}(\tau',n)}{\tau'-h(n)} \left\| \langle \tau-\tau'+\alpha\rangle^b \widehat{\eta}'(\tau-\tau'+\alpha) \right\|_{L^2_\tau}\, d\alpha\, d\tau' \right\|_{\ell^2_n} \\ &= \left\| \int_{R_1} \int_0^{\tau'-h(n)} \frac{\widehat{F}(\tau',n)}{\tau'-h(n)}\, d\alpha\, d\tau' \underbrace{\left\| \langle \tau\rangle^b \widehat{\eta}'(\tau) \right\|_{L^2_\tau}}_{<\infty} \right\|_{\ell^2_n}. \end{aligned}$$

Consequently, calculating the α integral and using Cauchy-Schwarz inequality in τ' gives

$$\begin{aligned} I &\lesssim_{\eta,b} \left\| \int_{R_1} \widehat{F}(\tau',n)\, d\tau' \right\|_{\ell^2_n} \\ &= \left\| \int_{R_1} \langle \tau'-h(n)\rangle^{b-1} \langle \tau'-h(n)\rangle^{-(b-1)} \widehat{F}(\tau',n)\, d\tau' \right\|_{\ell^2_n} \\ &\leq \left\| \underbrace{\left(\int_{R_1} \langle \tau'-h(n)\rangle^{-2(b-1)}\, d\tau' \right)^{\frac{1}{2}}}_{<\infty} \left(\int_{R_1} \langle \tau'-h(n)\rangle^{2(b-1)} \left|\widehat{F}(\tau',n)\right|^2\, d\tau' \right)^{\frac{1}{2}} \right\|_{\ell^2_n} \\ &\lesssim \|F\|_{X^{0,b-1}}. \end{aligned}$$

<u>Case 2:</u> $|\tau'-h(n)| \geq 1$

We start by focusing on the first term of the space-time Fourier transform. Note that $\langle \tau'-h(n)\rangle \sim |\tau'-h(n)|$. Then, using the fact that $|\tau-h(n)|^b \lesssim_b |\tau-\tau'|^b + |\tau'-h(n)|^b$,

for any $b > 0$,

$$
\begin{aligned}
II &:= \left\| \langle \tau - h(n) \rangle^b \int_{\mathbb{R}} \widehat{\eta}(\tau - \tau') \frac{1}{\tau' - h(n)} \widehat{F}(\tau', n) \, d\tau' \right\|_{\ell^2_n L^2_\tau(R_2)} \\
&\lesssim_b \left\| \int_{\mathbb{R}} \widehat{\eta}(\tau - \tau') \frac{\langle \tau - \tau' \rangle^b}{\tau' - h(n)} \widehat{F}(\tau', n) \, d\tau' \right\|_{\ell^2_n L^2_\tau(R_2)} \\
&\qquad + \left\| \int_{\mathbb{R}} \widehat{\eta}(\tau - \tau') \frac{\langle \tau' - h(n) \rangle^b}{\tau' - h(n)} \widehat{F}(\tau', n) \, d\tau' \right\|_{\ell^2_n L^2_\tau(R_2)} \\
&=: A + B.
\end{aligned}
$$

For the first term, use Minkowski's integral inequality in time and Cauchy-Schwarz

$$
\begin{aligned}
A &\leq \left\| \int_{\mathbb{R}} \widehat{\eta}(\tau - \tau') \frac{\langle \tau - \tau' \rangle^b}{\langle \tau' - h(n) \rangle} \widehat{F}(\tau', n) \, d\tau' \right\|_{\ell^2_n L^2_\tau(R_2)} \\
&\lesssim \left\| \int_{\mathbb{R}} \left\| \widehat{\eta}(\tau - \tau') \frac{\langle \tau - \tau' \rangle^b}{\langle \tau' - h(n) \rangle} \widehat{F}(\tau', n) \right\|_{L^2_\tau(R_2)} d\tau' \right\|_{\ell^2_n} \\
&= \left\| \int_{\mathbb{R}} \left| \frac{\langle \tau' - h(n) \rangle^{b-1}}{\langle \tau' - h(n) \rangle^b} \widehat{F}(\tau', n) \right| \|\eta\|_{H^b_t} \, d\tau' \right\|_{\ell^2_n} \\
&\lesssim_\eta \left\| \left(\int_{\mathbb{R}} \langle \tau' - h(n) \rangle^{-2b} \, d\tau' \right)^{\frac{1}{2}} \langle \tau' - h(n) \rangle^{b-1} \widehat{F}(\tau', n) \right\|_{\ell^2_n L^2_{\tau'}} \\
&\lesssim_b \|F\|_{X^{0,b-1}},
\end{aligned}
$$

since $b > \frac{1}{2}$.
For the second term,

$$
\begin{aligned}
B &\leq \left\| \int_{\mathbb{R}} \widehat{\eta}(\tau - \tau') \frac{\langle \tau' - h(n) \rangle^b}{\tau' - h(n)} \widehat{F}(\tau', n) \, d\tau' \right\|_{\ell^2_n L^2_\tau(R_2)} \\
&\lesssim \left\| \int_{\mathbb{R}} \left\| \widehat{\eta}(\tau - \tau') \frac{\langle \tau' - h(n) \rangle^b}{\tau' - h(n)} \widehat{F}(\tau', n) \right\|_{L^2_\tau(R_2)} d\tau' \right\|_{\ell^2_n} \\
&\lesssim \left\| \int_{\mathbb{R}} \left| \langle \tau' - h(n) \rangle^{b-1} \widehat{F}(\tau', n) \right|^2 \|\eta\|_{L^2_t} \, d\tau' \right\|_{\ell^2_n} \\
&\lesssim_\eta \|\eta\|_{L^2_t} \|F\|_{X^{0,b-1}}.
\end{aligned}
$$

Similarly, for the other component of the space-time Fourier Transform, we require $b > 1/2$, using Minkowski's integral inequality and Cauchy-Schwarz

$$\begin{aligned}
III &:= \left\| \langle \tau - h(n)\rangle^b \int_{\mathbb{R}} \widehat{\eta}(\tau - h(n)) \frac{1}{\tau' - h(n)} \widehat{F}(\tau', n)\, d\tau' \right\|_{\ell^2_n L^2_\tau(R_2)} \\
&\lesssim \left\| \int_{\mathbb{R}} \left\| \widehat{F}(\tau', n) \widehat{\eta}(\tau - h(n)) \frac{\langle \tau - h(n)\rangle^b}{\tau' - h(n)} \right\|_{L^2_\tau(R_2)} d\tau' \right\|_{\ell^2_n} \\
&= \left\| \int_{\mathbb{R}} \widehat{F}(\tau', n) \frac{1}{\tau' - h(n)} \|\eta\|_{H^b_t}\, d\tau' \right\|_{\ell^2_n} \\
&\lesssim_{\eta,b} \left\| \int_{\mathbb{R}} \widehat{F}(\tau', n) \frac{\langle \tau' - h(n)\rangle^{b-1}}{\langle \tau' - h(n)\rangle^b}\, d\tau' \right\|_{\ell^2_n} \\
&\leq \left\| \left(\int_{\mathbb{R}} \langle \tau' - h(n)\rangle^{-2b}\, d\tau' \right)^{\frac{1}{2}} \widehat{F}(\tau', n) \langle \tau' - h(n)\rangle^{b-1} \right\|_{\ell^2_n L^2_{\tau'}} \\
&\lesssim \|F\|_{X^{0,b}}.
\end{aligned}$$

□

Remark. Proposition 2.6 states that the time integration in $\mathcal{I}$ incurs in the loss of one time derivative in F.

2.4.3 Other Relevant Estimates

In order to complete the contraction mapping argument, we will require more control on the integral term in the Duhamel's formula. The following lemma takes advantage of the time cutoff, increasing time regularity at the cost of a small power of T.

Lemma 2.7 (Time Localisation Estimate)**.** *Let $\eta \in \mathcal{S}_t$ and $-\frac{1}{2} < b' \leq b < \frac{1}{2}$, then for any $0 < T < 1$ we have*

$$\|\eta\,(t/T)\, u\|_{X^{s,b'}} \lesssim_{\eta,b,b'} T^{b-b'} \|u\|_{X^{s,b}}. \tag{2.6}$$

Proof. We may take $s = 0$ without loss of generality, otherwise repeat the argument for $U := \langle \nabla \rangle^s u$. We only show the estimate for the case $0 = b' < b < \frac{1}{2}$, as the remaining parameter regions can be obtained by interpolation and duality.
Consider two regions, R_1 with $\langle \tau - h(n)\rangle \geq \frac{1}{T}$ and R_2 with $\langle \tau - h(n)\rangle \leq \frac{1}{T}$. Note that

$$\begin{aligned}
\|\widehat{u}(t,n)\|_{\ell^2_n} &= \left\| \int_{\mathbb{R}} \widehat{u}(\tau, n) e^{it\tau}\, d\tau \right\|_{\ell^2_n} \\
&\leq \left\| \int_{R_1} \widehat{u}(\tau, n) e^{it\tau}\, d\tau \right\|_{\ell^2_n} + \left\| \int_{R_2} \widehat{u}(\tau, n) e^{it\tau}\, d\tau \right\|_{\ell^2_n} =: I + II.
\end{aligned}$$

Then, in R_1

$$\begin{aligned}
I^2 &\leq \sum_{n \in \mathbb{Z}^d} \int_{R_1} \langle \tau - h(n)\rangle^{-2b} \langle \tau - h(n)\rangle^{2b} |\widehat{u}(\tau, n)|^2\, d\tau \\
&\leq \sum_{n \in \mathbb{Z}^d} \int_{R_1} T^{2b} \langle \tau - h(n)\rangle^{2b} |\widehat{u}(\tau, m)|^2\, d\tau \qquad \left(\langle \tau - h(n)\rangle^{-2b} \leq T^{2b} \right) \\
&\leq T^{2b} \|u\|^2_{X^{0,b}}.
\end{aligned}$$

Similarly, in R_2, using Cauchy-Schwarz inequality

$$\begin{aligned}
II^2 &= \sum_{n\in\mathbb{Z}^d} \left| \int_{R_2} \langle \tau - h(n)\rangle^b \langle \tau - h(n)\rangle^{-(b+\frac{1}{2})+\frac{1}{2}} \widehat{u}(\tau,n) e^{it\tau}\, d\tau \right|^2 \\
&\leq \sum_{n\in\mathbb{Z}^d} \left| \int_{R_2} \langle \tau - h(n)\rangle^b \langle \tau - h(n)\rangle^{-(b+\frac{1}{2})} T^{-\frac{1}{2}} \widehat{u}(\tau,n) e^{it\tau}\, d\tau \right|^2 \\
&\leq \sum_{n\in\mathbb{Z}^d} \left\{ \int_{R_2} \langle \tau - h(n)\rangle^{2b} \left|\widehat{u}(\tau,n)\right|^2\, d\tau \right\} \left\{ \int_{R_2} \langle \tau - h(n)\rangle^{-2(b+\frac{1}{2})} T^{-1}\, d\tau \right\} \\
&\leq T^{-1} T^{2(b+\frac{1}{2})-1} \|u\|^2_{X^{0,b}} = T^{2b-1} \|u\|^2_{X^{0,b}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\eta(t/T)u\|_{X^{0,0}} &= \|\mathcal{F}\left(\eta(t/T)u\right)(\tau,n)\|_{L^2_\tau \ell^2_n} \\
&= \|\eta(t/T)\widehat{u}(t,n)\|_{L^2_t \ell^2_n} \\
&= \left\|\eta(t/T)\|u(t)\|_{\ell^2_n}\right\|_{L^2_t} \\
&\leq \left\{T^b + T^{b-1/2}\right\} \|u\|_{X^{0,b}} \|\eta(t/T)\|_{L^2_t}.
\end{aligned}$$

Thus, to obtain the intended estimate we need to improve the bound using the norm of η,

$$\|\eta(t/T)\|^2_{L^2_t} = \int_{\mathbb{R}} |\eta(t/T)|^2\, dt = \int_{\mathbb{R}} |\eta(t')|^2 T\, dt' = T \underbrace{\|\eta(t)\|^2_{L^2_t}}_{c_1<\infty},$$

which implies

$$\|\eta(t/T)u\|_{X^{0,0}} \lesssim_\eta \left\{T^{b+\frac{1}{2}} + T^b\right\} \|u\|_{X^{0,b}} \leq 2T^b \|u\|_{X^{0,b}},$$

since $0 < T < 1$.

□

The following lemma allows us to 'transfer' estimates in H^s to estimates in $X^{s,b}$.

Lemma 2.8 (Transference Principle)**.** *Let $b > \frac{1}{2}$ and $Y = L^q_t L^r_x$ or $Y = L^r_x L^q_t$, for $1 \leq p, q < \infty$. Let $\mathcal{T}$ a k-linear operator such that*

$$\left\|\mathcal{T}\left(e^{t\mathcal{L}} f_1, \dots, e^{t\mathcal{L}} f_k\right)\right\|_Y \lesssim \prod_{j=1}^k \|f_j\|_{H^{s_j}},$$

for all $f_j \in H^{s_j}$, $1 \leq j \leq k$. Then

$$\|\mathcal{T}\left(u_1, \dots, u_k\right)\|_Y \lesssim \prod_{j=1}^k \|u_j\|_{X^{s_j,b}},$$

for all $u_j \in X^{s_j,b}$, $1 \leq j \leq k$.

Proof. Let $1 \leq j \leq k$, $\tau = \tau_j + h(n)$ and

$$\begin{aligned} f_{j,\tau_j}(x) :&= \sum_{n\in\mathbb{Z}^d} \widehat{u_j}(\tau_j + h(n), n) e^{ix\cdot n} \\ \implies e^{t\mathcal{L}} f_{j,\tau_j} &= \sum_{n\in\mathbb{Z}^d} \widehat{u_j}(\tau_j + h(n), n) e^{ix\cdot n + ith(n)} \\ \implies u_j(t,x) &= \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}^d} \widehat{u_j}(\tau, n) e^{ix\cdot n + it\tau}\, d\tau = \int_{\mathbb{R}} e^{it\tau_j} e^{t\mathcal{L}} f_{j,\tau_j}\, d\tau_j. \end{aligned}$$

Then, using the k-linearity of operator $\mathcal{T}$,

$$\begin{aligned} \mathcal{T}(u_1, \dots, u_k) &= \mathcal{T}\left(\int_{\mathbb{R}} e^{it\tau_1} e^{t\mathcal{L}} f_{1,\tau_1}\, d\tau_1, \dots, \int_{\mathbb{R}} e^{it\tau_k} e^{t\mathcal{L}} f_{k,\tau_k}\, d\tau_k \right) \\ &= \int_{\mathbb{R}^k} e^{it\tau_1} \dots e^{it\tau_k}\, \mathcal{T}\left(e^{t\mathcal{L}} f_{1,\tau_1}, \dots, e^{t\mathcal{L}} f_{k,\tau_k} \right) d\tau_1 \dots d\tau_k, \end{aligned}$$

and calculating the Y-norm gives, using Minkowki's integral inequality,

$$\begin{aligned} \|\mathcal{T}(u_1, \dots, u_k)\|_Y &= \left\| \int_{\mathbb{R}^k} e^{it\tau_1} \dots e^{it\tau_k}\, \mathcal{T}\left(e^{t\mathcal{L}} f_{1,\tau_1}, \dots, e^{t\mathcal{L}} f_{k,\tau_k} \right) d\tau_1 \dots d\tau_k \right\|_Y \\ &\lesssim \int_{\mathbb{R}^k} |e^{it\tau_1} \dots e^{it\tau_k}| \left\| \mathcal{T}\left(e^{t\mathcal{L}} f_{1,\tau_1}, \dots, e^{t\mathcal{L}} f_{k,\tau_k} \right) \right\|_Y d\tau_1 \dots d\tau_k \\ &\lesssim \int_{\mathbb{R}^k} \prod_{j=1}^k \left\| f_{j,\tau_j} \right\|_{H^{s_j}} d\tau_1 \dots d\tau_k. \end{aligned}$$

Applying Cauchy-Schwarz inequality and using the assumption,

$$\begin{aligned} \|\mathcal{T}(u_1, \dots, u_k)\|_Y &= \prod_{j=1}^k \int_{\mathbb{R}} \langle \tau_j \rangle^{-b} \langle \tau_j \rangle^b \left\| f_{j,\tau_j} \right\|_{H^{s_j}} d\tau_j \\ &\leq \prod_{j=1}^k \underbrace{\left\{ \int_{\mathbb{R}} \langle \tau_j \rangle^{-2b}\, d\tau_j \right\}^{\frac{1}{2}}}_{<\infty \text{ as } b>1/2} \left\{ \int_{\mathbb{R}} \langle \tau_j \rangle^{2b} \left\| f_{j,\tau_j} \right\|_{H^{s_j}}^2 d\tau_j \right\}^{\frac{1}{2}} \\ &\lesssim_b \prod_{j=1}^k \left\| \langle \tau_j \rangle^b \widehat{u_j}(\tau_j + h(n), n) \right\|_{L^2_{\tau_j} \ell^2_n} \\ &= \prod_{j=1}^k \left\| \langle \tau - h(n) \rangle^b \widehat{u_j}(\tau, n) \right\|_{L^2_\tau \ell^2_n} = \prod_{j=1}^k \|u\|_{X^{s_j,b}}. \end{aligned}$$

□

2.5 Nonlinear Equation

We are interested in nonlinear equations, thus we can substitute the forcing F in (2.1) by a nonlinear term $\mathcal{N}(u)$

$$\begin{cases} \partial_t u(t,x) = \mathcal{L}u(t,x) + \mathcal{N}(u)(t,x) \\ u(0,x) = u_0(x) \end{cases}, u : \mathbb{R} \times \mathbb{T}^d \to \mathbb{K}, \tag{2.7}$$

with $\mathcal{N}$ a smooth function.

Remark. For NLS and KdV type equations, we will consider

$$\begin{aligned} \text{NLS}: &\quad \mathcal{N}(u) = \pm|u|^{p-1}u, \\ \text{KdV}: &\quad \mathcal{N}(u) = \pm\partial_x\big(u^k\big), \end{aligned}$$

with $p \geq 1$ odd integer (to guarantee an algebraic nonlinearity) and $k \in \mathbb{Z}_{>0}$.

Using Duhamel's formula (2.2), with the nonlinearity instead of the forcing F, smooth time cutoff $\eta \in \mathcal{S}_t$ and $0 < T < 1$, defines the *solution map* Γ_{u_0} for the nonlinear equation (2.7),

$$\Gamma_{u_0}(u)(t,x) := \eta(t)e^{t\mathcal{L}}u_0(x) + \eta\left(\frac{t}{T}\right)\int_0^t e^{(t-t')\mathcal{L}}\mathcal{N}(u)(t',x)\,dt'.$$

Thus, solving (2.7) reduces to finding a fixed point for Γ_{u_0}.

The contraction mapping argument relies on the idea of perturbing the linear solution, while expecting it to remain on a neighbourhood of the linear solution for a short time, after time localisation. As seen before, the space-time Fourier Transfom of the linear solution is supported on the hypersurface given by $\tau = h(n)$, therefore we expect the counterpart for the time-localised nonlinear solution to remain supported on a neighbourhood of the hypersurface, as illustrated on Figure 2.1.

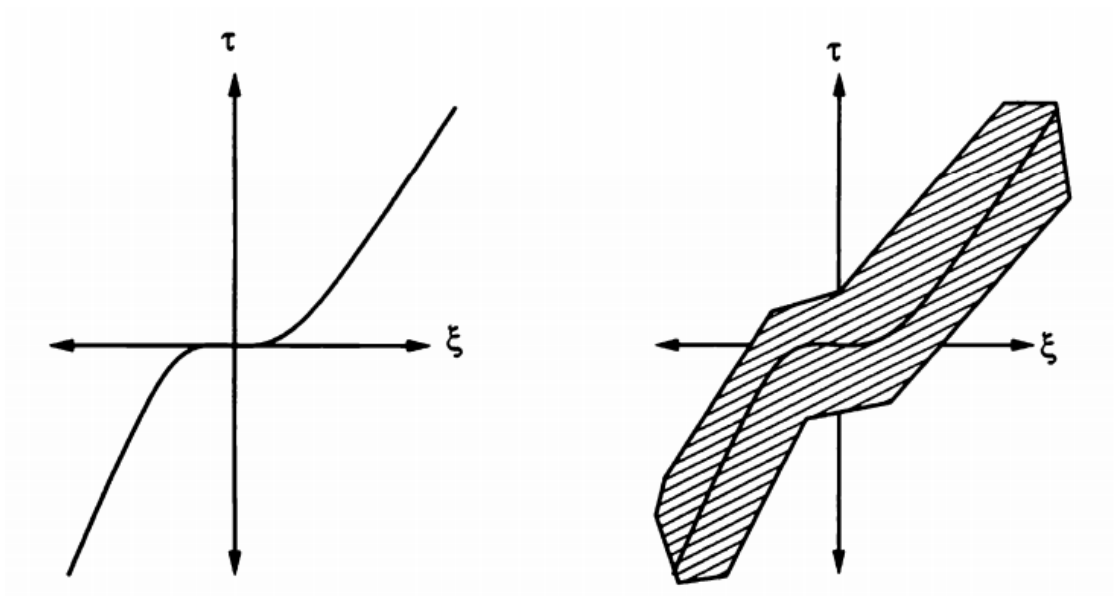

FIGURE 2.1: Illustration of support of space-time Fourier transform for linear and nonlinear solution of KdV. Adapted from [9].

In summary, the $X^{s,b}$ spaces capture this behaviour for the nonlinear solution and the homogeneous and inhomogeneous linear estimates can still be applied in the presence of the nonlinearity.

3 Nonlinear Schrödinger Equation

In this chapter, we will consider the one dimensional nonlinear Schrödinger equation, with cubic and quintic nonlinearities, in the periodic setting. In order to complete the contraction mapping argument, we need nonlinear estimates to bound the integral part of the solution map

$$\Gamma_{u_0}(u)(t,x) = \eta(t)S(t)u_0(x) \pm i\eta(t/T)\int_0^t S(t-t')|u|^{p-1}u(t')\,dt',$$

after application of the inhomogeneous linear estimate. Note that the linear propagator $S(t)$ is given by $\mathcal{F}\left\{S(t)f(x)\right\}(n) = e^{-itn^2}\widehat{f}(n)$.

Firstly, we require Strichartz estimates. Heuristically, considering a duality argument, we are interested in controlling

$$\sup_{\|v\|_{L^2_{t,x}}\leq 1}\int_{\mathbb{R}}\int_{\mathbb{T}} u^p v\,dx\,dt \leq \sup_{\|v\|_{L^2_{t,x}}\leq 1}\|u\|^p_{L^{p+1}_{t,x}}\|v\|_{L^{p+1}_{t,x}}.$$

Thus, for cubic and quintic NLS, we want L^4 and L^6 Strichartz, respectively. These estimates, due to Bourgain, allowed for an improvement in LWP results.

Secondly, using Strichartz estimates, we prove the trilinear and quintilinear estimates, the last results neede to complete the proof of local well-posedness.

Lastly, we will complete the contraction mapping argument.

Note that these two examples were chosen for their simplicity, but the results display techniques which can be adapted to regularities of different order or higher order dimensions.

The interested reader should refer to [2, 3] for more details.

3.1 Strichartz Estimates

On the real line, Strichartz estimates exploit dispersion relations, which are not available in the periodic setting. Nevertheless, on $\mathbb{T}^d$ there are some Strichartz estimates available, namely of $L^4_{t,x}$ and $L^6_{t,x}$ type, which rely on a space-time Fourier analytic approach. Such estimates are required to control the nonlinearity, as one wants to move from spaces of higher integrability to L^2 based spaces.

As an historical note, in 1974, Zygmund proved a version of L^4-Strichartz estimates in one dimension, stated below.

Proposition 3.1 (Zygmund '74, [10])**.** *Let $S(t)$ be the linear propagator for* (NLS) *and $u_0 \in L^2_{t,x}(\mathbb{T}^2)$, then*

$$\|S(t)u_0\|_{L^4_{t,x}(\mathbb{T}^2)} \lesssim \|u_0\|_{L^2_{t,x}(\mathbb{T}^2)}.$$

Using Transference Principle (Lemma 2.8), one obtains the following estimate, which requires periodicity in time, as well.

Proposition 3.2. *The following estimate holds for $b > \frac{1}{2}$,*

$$\|u\|_{L^4_{t,x}(\mathbb{T}^2)} \lesssim \|u\|_{X^{0,b}}.$$

Bourgain, in 1993, contributed with a sharper L^4-Strichartz in one dimension, which lead to a great improvement in local well-posedness results for (NLS). This result was first proposed in the periodic setting, both in time and space, but a simple modification to the argument allows for a global-in-time result.

Proposition 3.3 (Bourgain '93, [2])**.** *The following estimate holds*

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0,\frac{3}{8}}}.$$

Note that this result is a great refinement in terms of time regularity and, most importantly, arrives at a space where $0 < b < \frac{1}{2}$. Heuristically, one can see the sharpness in $b = \frac{3}{8}$. Applying Sobolev inequality gives $X^{\frac{1}{4},\frac{1}{4}} \subset L^4_{t,x}(\mathbb{R}\times\mathbb{T})$. Looking at the equation, two spatial derivatives "correspond" to one time derivative, thus formally moving the space derivatives to the time variable gives $b = \frac{1}{4} + \frac{1}{2}\frac{1}{4} = \frac{3}{8}$.

Proof of Proposition 3.3. The proof follows an argument by Nikolay Tzvetkov.
Let $u = \sum_M u_M$ with M dyadic and u_M the portion of u localised on the space-time frequency region $M \leq \langle \tau + n^2 \rangle < 2M$ on the Fourier side. Then,

$$\begin{aligned}
\sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}} &= \sum_M M^{\frac{3}{4}} \|\widehat{u_M}\|^2_{\ell^2_n L^2_\tau} \\
&= \sum_M M^{\frac{3}{4}} \int_{\mathbb{R}} \sum_n |\widehat{u_M}(\tau, n)|^2 \, d\tau \\
&\leq \sum_M \int_{\mathbb{R}} \sum_n \left(\langle \tau + n^2 \rangle\right)^{\frac{3}{4}} |\widehat{u_M}(\tau, n)|^2 \, d\tau \\
&= \sum_M \|u_M\|^2_{X^{0,\frac{3}{8}}} \\
&\sim \|u\|^2_{X^{0,\frac{3}{8}}}.
\end{aligned}$$

Now we want to bound $\|u\|_{L^4_{t,x}}$ from above by the left-hand side of the previous inequality. Note that, using Minkowski's discrete inequality,

$$\begin{aligned}
\|u\|^2_{L^4_{t,x}} &= \left(\int_{\mathbb{R}} \int_{\mathbb{T}} |u|^2 |u|^2 \, dx \, dt \right)^{1/2} \\
&= \left(\int_{\mathbb{R}} \int_{\mathbb{T}} \left(\sum_M |u_M|^2 \right) \left(\sum_{M'} |u_{M'}|^2 \right) \, dx \, dt \right)^{1/2} \\
&\lesssim \left(\int_{\mathbb{R}} \int_{\mathbb{T}} \sum_M \sum_{M \leq M'} |u_M|^2 |u_{M'}|^2 \, dx \, dt \right)^{1/2} \\
&= \left(\sum_M \sum_{M \leq M'} \int_{\mathbb{R}} \int_{\mathbb{T}} |u_M|^2 |u_{M'}|^2 \, dx \, dt \right)^{1/2} \\
&= \left(\sum_M \sum_{M \leq M'} \|u_M u_{M'}\|^2_{L^2_{t,x}} \right)^{1/2} \\
&\leq \sum_M \sum_{M \leq M'} \|u_M u_{M'}\|_{L^2_{t,x}}.
\end{aligned}$$

Thus, it suffices to show

$$\sum_M \sum_{M \leq M'} \|u_M u_{M'}\|_{L^2_{t,x}} \lesssim \sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}}.$$

Since M, M' are dyadic, we can write $M' = 2^k M$, for some $k \in \mathbb{Z}_{\geq 0}$. Then, if for some $\varepsilon > 0$

$$\sum_M \|u_M u_{2^k M}\|_{L^2_{t,x}} \lesssim 2^{-\varepsilon k} \sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}},$$

it follows that

$$\sum_M \sum_{k \geq 0} \|u_M u_{2^k M}\|_{L^2_{t,x}} \lesssim \underbrace{\left(\sum_{k \geq 0} 2^{-\varepsilon k} \right)}_{<\infty} \sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}}.$$

Consequently, we can focus on proving

$$\|u_M u_{2^k M}\|_{L^2_{t,x}} \lesssim 2^{-\varepsilon k} M^{\frac{3}{8}} \|u_M\|_{L^2_{t,x}} \left(2^k M\right)^{\frac{3}{8}} \|u_{2^k M}\|_{L^2_{t,x}}, \tag{3.1}$$

since when summing in M it gives the following, using Cauchy-Schwarz inequality,

$$\begin{aligned}
\sum_M \|u_M u_{2^k M}\|_{L^2_{t,x}} &\lesssim \sum_M 2^{-\varepsilon k} M^{\frac{3}{8}} \|u_M\|_{L^2_{t,x}} \left(2^k M\right)^{\frac{3}{8}} \|u_{2^k M}\|_{L^2_{t,x}} \\
&\leq 2^{-\varepsilon k} \left(\sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}} \right)^{1/2} \left(\sum_M (2^k M)^{\frac{3}{4}} \|u_{2^k M}\|^2_{L^2_{t,x}} \right)^{1/2} \\
&\leq 2^{-\varepsilon k} \sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}}.
\end{aligned}$$

In order to prove (3.1), we start by normalizing u_M and $u_{2^k M}$. Let

$$v_M = \frac{u_M}{\|u_M\|_{L^2_{t,x}}}, \quad v_{2^k M} = \frac{u_{2^k M}}{\|u_{2^k M}\|_{L^2_{t,x}}},$$

which simplifies (3.1) to

$$\|v_M v_{2^k M}\|_{L^2_{t,x}} \lesssim 2^{-\varepsilon k + \frac{3}{8} k} M^{\frac{3}{4}}.$$

Using Plancherel's identity, it suffices to show

$$\left\| \sum_{n_1 + n_1 = n} \int_{\tau_1 + \tau_2 = \tau} \widehat{v_M}(\tau_1, n_1) \widehat{v_{2^k M}}(\tau_2, n_2) \, d\tau_1 \right\|_{\ell^2_n L^2_\tau} \lesssim 2^{(\frac{3}{8} - \varepsilon) k} M^{\frac{3}{4}}.$$

Using Cauchy-Schwarz on the left-hand side, taking into account the supports of $\widehat{v_M}$ and $\widehat{v_{2^k M}}$,

$$\begin{aligned}
&\left\| \Big(\sum_{n_1 + n_2 = n} \int_* 1 \, d\tau_1 \Big)^{1/2} \Big(\sum_{n_1 + n_2 = n} \int_{\tau_1 + \tau_2 = \tau} |\widehat{v_M}(\tau_1, n_1) \widehat{v_{2^k M}}(\tau_2, n_2)|^2 \, d\tau_1 \Big)^{1/2} \right\|_{\ell^2_n L^2_\tau} \\
&\quad \leq \sup_{n, \tau} \Big(\sum_{n_1 + n_2 = n} \int_* 1 \, d\tau_1 \Big)^{1/2} \left\| \sum_{n_1 + n_2 = n} \int_{\tau_1 + \tau_2 = \tau} |\widehat{v_M}(\tau_1, n_1) \widehat{v_{2^k M}}(\tau_2, n_2)|^2 \, d\tau_1 \right\|_{\ell^1_n L^1_\tau},
\end{aligned}$$

with $* := \big\{(\tau_1, \tau_2) : \ \tau = \tau_1 + \tau_2, \ \tau_1 = -n_1^2 + \mathcal{O}(M), \ \tau_2 = -n_2^2 + \mathcal{O}(2^k M)\big\}$.

Note that the second term is equal to one because of the normalization, using Fubini's theorem

$$\begin{aligned}
&\Big\| \sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} |\widehat{v_M}(\tau_1, n_1)\widehat{v_{2^k M}}(\tau_2, n_2)|^2 \, d\tau_1 \Big\|_{\ell^1_n L^1_\tau} \\
&= \Big(\sum_{n_1} \int_{\mathbb{R}} |\widehat{v_M}(\tau_1, n_1)|^2 \|v_{2^k M}\|^2_{\ell^2_n L^2_\tau} \, d\tau_1 \Big)^{1/2} \\
&= \|v_M\|_{\ell^2_{n_1} L^2_{\tau_1}} \|v_{2^k M}\|_{\ell^2_n L^2_\tau} = 1.
\end{aligned}$$

Thus, it suffices to show that

$$\sum_{n_1+n_2=n} \int\limits_{\substack{\tau_1+\tau_2=\tau \\ \tau_1=-n_1^2+\mathcal{O}(M) \\ \tau_2=-n_2^2+\mathcal{O}(2^k M)}} 1 \, d\tau_1 \lesssim 2^{(\frac{3}{4}-2\varepsilon)k} M^{\frac{3}{2}}, \tag{3.2}$$

which is a counting problem. Fixing n, τ, for the integral to be nonzero, we must have τ_1, τ_2, n_1, n_2 which satisfy

$$\begin{cases} \tau = \tau_1 + \tau_2 = -n_1^2 - n_2^2 + \mathcal{O}(M) + \mathcal{O}(2^k M) \\ n = n_1 + n_2 \end{cases}.$$

Then, $\tau_1 = \tau - \tau_2 = \tau + n_2^2 + \mathcal{O}(2^k M) = -n_1^2 + \mathcal{O}(M)$. When focusing on the integral we are only concerned with the length of the interval where τ_1 can move, from the previous calculation it is clear that it is $\mathcal{O}(M)$.
Now, focusing on the sum, it remains to show that

$$\sum_{\substack{n_1+n_2=n \\ \tau=-n_1^2-n_2^2+\mathcal{O}(2^k M)}} 1 \lesssim 2^{(\frac{3}{4}-2\varepsilon)k} M^{\frac{1}{2}}.$$

Note that

$$(n_1 - n_2)^2 = n_1^2 + n_2^2 - 2n_1 n_2 = 2(n_1^2 + n_2^2) - (n_1 + n_2)^2 = -2\tau - n^2 + \mathcal{O}(2^k M).$$

Since τ, n are fixed, the number of solutions $(n_1 - n_2)$ is at most of order $\mathcal{O}(2^{\frac{k}{2}} M^{\frac{1}{2}})$. Thus,

$$\sum_{\substack{n_1+n_2=n \\ \tau=-n_1^2-n_2^2+\mathcal{O}(2^k M)}} 1 \lesssim 2^{\frac{k}{2}} M^{\frac{1}{2}}.$$

Choosing $\varepsilon > 0$ as follows

$$\frac{3}{4} - 2\varepsilon = \frac{1}{2} \implies \varepsilon = \frac{1}{8},$$

gives the intended result. □

L^4-Strichartz as stated is not useful to control a quintic nonlinearity. Therefore, Bourgain proposed L^6-Strichartz, which are much weaker, as one cannot get rid of the power N^{0+} in the following proposition, preventing an estimate in $X^{0,b}$ as for L^4.[1]

[1] The notation $N^{\alpha+}$ represents $N^{\alpha+\varepsilon}$ for some small $0 < \varepsilon \ll 1$.

Proposition 3.4 (Bourgain '93, [2]). *Let $u_0 \in L^2_x(\mathbb{T})$ and $\eta \in \mathcal{S}_t(\mathbb{R})$ a smooth time cutoff, then*

$$\|\eta(t)S(t)P_{\leq N}u_0\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim N^{0+}\|P_{\leq N}u_0\|_{L^2_x(\mathbb{T})},$$

with $P_{\leq N}$ denoting the Dirichlet projection onto $|n| \leq N$.

Proof. We start by calculating the space-time Fourier Transform of the function on the left-hand side. Note that,

$$\begin{aligned}
\eta(t)\sum_{|n|\leq N} e^{i(xn-tn^2)}\widehat{f}(n) &= \eta(t)\sum_n e^{i(nx-tn^2)}\widehat{f}(n)\mathbb{1}_{(|n|\leq N)} \\
&= \eta(t)\sum_n e^{inx}\left(e^{-itn^2}\widehat{f}(n)\mathbb{1}_{(|n|\leq N)}\right) \\
&= \eta(t)\mathcal{F}^{-1}\left(e^{-itn^2}\widehat{f}(n)\mathbb{1}_{(|n|\leq N)}\right)(t,x),
\end{aligned}$$

then,

$$\begin{aligned}
\mathcal{F}\left(\eta(t)\sum_{|n|\leq N} e^{i(xn-tn^2)}\widehat{f}(n)\right)(t,n) &= \eta(t)e^{-itn^2}\widehat{f}(n)\mathbb{1}_{(|n|\leq N)}, \\
\mathcal{F}\left(\eta(t)\sum_{|n|\leq N} e^{i(xn-tn^2)}\widehat{f}(n)\right)(\tau,n) &= \int_{\mathbb{R}} \eta(t)e^{-itn^2}\widehat{f}(n)\mathbb{1}_{(|n|\leq N)}e^{-it\tau}\,dt \\
&= \widehat{f}(n)\widehat{\eta}(\tau+n^2)\mathbb{1}_{(|n|\leq N)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|\eta(t)S(t)P_{\leq N}f\|^6_{L^6_{t,x}(\mathbb{R}\times\mathbb{T})} \\
&\quad= \left\|(\eta(t)S(t)P_{\leq N}f)^3\right\|^2_{L^2_{t,x}(\mathbb{R}\times\mathbb{T})} \\
&\quad= \left\|\sum_{n=n_1-n_2+n_3}\int_{\tau=\tau_1-\tau_2+\tau_3}\prod_{j=1}^3\left\{\widehat{f}(n_j)\widehat{\eta}(\tau_j+n_j^2)\mathbb{1}_{(|n_j|\leq N)}\right\}\,d\tau_1\,d\tau_2\right\|^2_{L^2_{\tau,n}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\tau=\tau_1-\tau_2+\tau_3}\prod_{j=1}^3\widehat{\eta}(\tau_j+n_j^2)\,d\tau_1\,d\tau_2 \\
&\quad= \int_{\tau=\tau_1-\tau_2+\tau_3}\widehat{\eta}(\tau_1+n_1^2)\widehat{\eta}(\tau_2+n_2^2)\widehat{\eta}(\tau-\tau_1+\tau_2-n_3^2)\,d\tau_1\,d\tau_2 \\
&\quad= \int\widehat{\eta}(\tau_1+n_1^2)\psi(\tau-\tau_1+n_2^2+n_3^2)\,d\tau_1 \\
&\quad= \tilde{\psi}(\tau+n_1^2-n_2^2+n_3^2),
\end{aligned}$$

for some $\psi, \tilde{\psi} \in \mathcal{S}_t(\mathbb{R})$.
Thus, it suffices to bound

$$\left\|\sum_{n=n_1-n_2+n_3}\prod_{j=1}^3\left\{\widehat{f}(n_j)\mathbb{1}_{(|n_j|\leq N)}\right\}\tilde{\psi}(\tau+n_1^2-n_2^2+n_3^2)\right\|^2_{\ell^2_n L^2_\tau} = (*).$$

We will consider two different cases.

<u>Case 1:</u> $n_1 = n \implies n_2 = n_3$
Note that as a consequence of symmetry, $n_3 = n$ is equivalent.

$$\begin{aligned}
(*) &= \left\| \sum_{n_2} \widehat{f}(n)\widehat{f}(n_2)\widehat{f}(n_2)\mathbb{1}_{(|n|,|n_2|\le N)}\tilde{\psi}(\tau+n^2)\right\|^2_{\ell^2_n L^2_\tau} \\
&= \left\| \sum_{n_2} \widehat{f}(n)\widehat{f}(n_2)\widehat{f}(n_2)\mathbb{1}_{(|n|,|n_2|\le N)}\|\tilde{\psi}\|_{L^2_\tau}\right\|^2_{\ell^2_n} \\
&\le \left(\sum_{|n_2|\le N} |\widehat{f}(n_2)|^2\right)^2 \underbrace{\|\tilde{\psi}\|^2_{L^2_\tau}}_{=c_1<\infty} \|\widehat{f}\mathbb{1}_{|n|\le N}\|^2_{\ell^2_n} \\
&= c_1\|P_{\le N} f\|^6_{L^2_x}.
\end{aligned}$$

<u>Case 2:</u> $n_1, n_3 \ne n$
Using Hölder's inequality,

$$\begin{aligned}
(*) &= \left\| \left\{ \sum_{n=n_1-n_2+n_3} \prod_{j=1}^{3} \left\{\widehat{f}(n_j)\mathbb{1}_{(|n_j|\le N)}\right\} \tilde{\psi}(\tau+n_1^2-n_2^2+n_3^2)\right\}^2\right\|_{\ell^1_n L^1_\tau} \\
&\lesssim \left\| \sum_{n=n_1-n_2+n_3} \prod_{j=1}^{3} |\widehat{f}(n_j)|^2\mathbb{1}_{(|n_j|\le N)}|\tilde{\psi}(\tau+n_1^2-n_2^2+n_3^2)|^2\right\|_{L^1_\tau \ell^1_n} \\
&\lesssim \left\| \sum_{*} \prod_{j=1}^{3} |\widehat{f}(n_j)|^2|\tilde{\psi}(\tau+n_1^2-n_2^2+n_3^2)|\right\|_{\ell^1_n L^1_\tau} \left\|\tilde{\psi}(\tau+n_1^2-n_2^2+n_3^2)\right\|_{\ell^\infty_n L^\infty_\tau} \\
&=: A_1 \cdot A_2, \qquad (3.3)
\end{aligned}$$

with the sum over the $* := \left\{(n_1,n_2)\in\mathbb{Z}^2 :\ n = n_1-n_2+n_3,\ |n_j|\le N, j=1,2,3\right\}$.
Thus,

$$\begin{aligned}
A_1 &= \underbrace{\|\tilde{\psi}\|_{L^1_\tau}}_{=c_2<\infty} \sum_n \sum_* \left(\prod_{j=1}^{3} |\widehat{f}(n_j)|^2\right) \\
&\le c_2 \sum_{|n_j|\le N} \left(\prod_{j=1}^{3} |\widehat{f}(n_j)|^2\right) \\
&= c_2\|P_{\le N} f\|^6_{L^2_x}.
\end{aligned}$$

Now, for the second term,

$$\begin{aligned}
A_2 &= \sup_{n,\tau} \left|\sum_* \tilde{\psi}(\tau+n_1^2-n_2^2+n_3^2)\right| \\
&\le \sup_{n,\tau} \sum_* |\tilde{\psi}(\tau+n_1^2-n_2^2+n_3^2)| \\
&\sim \sup_{n,\tau} \#\left\{(n_1,n_2)\in\mathbb{Z}^2 :\ n = n_1-n_2+n_3, |n_j|\le N, j=1,2,3\right\}.
\end{aligned}$$

Note that $\tau + n_1^2 - n_2^2 + n_3^2 = \mathcal{O}(1)$ and

$$\begin{aligned}
2(n_2 - n_1)(n_2 - n_3) &= (2n_2 - n_1 - n_3)^2 - (n_2 - n_1)^2 - (n_2 - n_3)^2 \\
&= (n_2 - n)^2 - (n_3 - n)^2 - (n_1 - n)^2 \\
&= -n^2 + 2n(n_1 - n_2 + n_3) + n_2^2 - n_1^2 - n_3^2 \\
&= n^2 - n_1^2 + n_2^2 - n_3^2 \\
&= n^2 + \tau + \mathcal{O}(1) =: c_{n,\tau}.
\end{aligned}$$

In addition, $|n| \leq 3N$ and $|\tau| \lesssim N^2$, as a consequence of the bound on n_j, $j = 1, 2, 3$. Then, given $n \in \mathbb{Z}$, $\tau \in \mathbb{R}$, there exists at most $\mathcal{O}(1)$ integers k such that $|k - c_{n,\tau}| \lesssim 1$, which implies

$$|k| = |n^2 + \tau| + \mathcal{O}(1) \lesssim N^2.$$

By divisor counting, there exist at most N^{0+} many

$$(n_2 - n_1)|k \quad (n_2 - n_3)|k.$$

Since $(n_2 - n_1) = (n_3 - n)$ and $(n_2 - n_3) = (n_1 - n)$, there are at most $\mathcal{O}(N^{0+})$ many n_1, n_3, which uniquely determine n_2, showing that $A_2 \lesssim N^{0+}$.

Combining both cases gives the intended result. □

We can use the Transference Principle (Lemma 2.8) to obtain a bound on the L^6-norm of ηu.

Corollary 3.5. *There exists small $\varepsilon > 0$, such that for $b > \frac{1}{2}$ the following estimate holds*

$$\|\eta(t)u\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{\varepsilon,b}}. \tag{3.4}$$

Proof. Let P_N represent the Dirichlet projection to $|n| \sim N$, with N dyadic. Using Proposition 3.4, there exists $\varepsilon > 0$ such that

$$\|\eta(t)S(t)P_N u_0\|_{L^6_{t,x}} = \|\eta(t)S(t)P_{\leq N}(P_N u_0)\|_{L^6_{t,x}} \lesssim N^{\varepsilon}\|P_{\leq N}(P_N u_0)\|_{L^2_x} = N^{\varepsilon}\|P_N u_0\|_{L^2_x}.$$

Then,

$$\begin{aligned}
\|\eta(t)S(t)u_0\|_{L^6_{t,x}} &= \|\eta(t)S(t)\sum_N P_N u_0\|_{L^6_{t,x}} \\
&\lesssim \sum_N \|\eta(t)S(t)P_N u_0\|_{L^6_{t,x}} \\
&\lesssim \sum_N N^{\varepsilon}\|P_N u_0\|_{L^2_x} \\
&= \sum_N N^{-\varepsilon}N^{2\varepsilon}\|P_N u_0\|_{L^2_x} \\
&\leq \left(\sum_N N^{-2\varepsilon}\right)^{1/2}\left(\sum_N N^{4\varepsilon}\|P_N u_0\|^2_{L^2_x}\right)^{1/2} \\
&\lesssim \|u_0\|_{H^{2\varepsilon}}.
\end{aligned}$$

Now, applying Transference Principle (Lemma 2.8), we obtain

$$\|\eta(t)u\|_{L^6_{t,x}} \lesssim \|u\|_{X^{2\varepsilon,b}},$$

for $b > \frac{1}{2}$, as intended. □

Before we established L^6-Strichartz estimates (3.4), for some $\alpha > 0$,

$$\|\eta(t)P_N u\|_{L^6(\mathbb{R}\times\mathbb{T})} \lesssim \|P_N u\|_{X^{\varepsilon,\frac{1}{2}+\alpha}}.$$

However, we need to use the Time Localisation estimate (Lemma 2.7) to gain control on the nonlinear part through the time of existence T. This estimate arrives in $X^{s,b}$ with $b > \frac{1}{2}$, so we need to create some room in the time regularity. To lower time regularity we can interpolate L^6-Strichartz and Sobolev inequality in time,

$$\|\eta(t)P_N u\|_{L^6_{t,x}} \lesssim N^{1/3}\|P_N u\|_{X^{0,\frac{1}{3}}}.$$

Interpolating the two results, we obtain

$$\|\eta(t)P_N u\|_{L^6_{t,x}} \lesssim N^{\beta}\|P_N u\|_{X^{0,\frac{1}{2}-\beta}}, \tag{3.5}$$

for some small enough $0 < \beta \ll 1$.

3.2 Cubic NLS

In this section, we focus on the 1-d cubic (NLS). In order to show LWP, we require homogeneous and inhomogeneous linear estimates, and L^4-Strichartz to show a nonlinear estimate. Note that the time regularity with Strichartz estimates ($b = \frac{3}{8} < \frac{1}{2}$), when applied to the nonlinearity will give some room to apply the time localisation estimate, giving control over the nonlinear part through the time of existence T.

The L^4-Strichartz estimate determined previously (Proposition 3.3) allows us to prove the following trilinear estimate.

Lemma 3.6. *For any* $0 < \varepsilon, \delta \leq \frac{1}{8}$,

$$\|u_1\overline{u_2}u_3\|_{X^{0,-\frac{1}{2}+\varepsilon}} \lesssim \prod_{j=1}^{3}\|u_j\|_{X^{0,\frac{1}{2}-\delta}}. \tag{3.6}$$

Proof. Using a duality argument, Hölder's inequality and $X^{s,b}$ embeddings,

$$\begin{aligned}
\|u_1\overline{u_2}u_3\|_{X^{0,-\frac{1}{2}+\varepsilon}} &= \sup_{\|v\|_{X^{0,\frac{1}{2}-\varepsilon}}\leq 1}\left|\int_{\mathbb{R}}\int_{\mathbb{T}} u_1\overline{u_2}u_3\overline{v}\,dx\,dt\right| \\
&\lesssim \sup\prod_{j=1}^{3}\|u_j\|_{L^4_{t,x}}\|v\|_{L^4_{t,x}} \\
&\lesssim \sup\prod_{j=1}^{3}\|u_j\|_{X^{0,\frac{3}{8}}}\|v\|_{X^{0,\frac{3}{8}}} \\
&\leq \sup\prod_{j=1}^{3}\|u_j\|_{X^{0,\frac{1}{2}-\delta}}\|v\|_{X^{0,\frac{1}{2}-\varepsilon}} \\
&\leq \prod_{j=1}^{3}\|u_j\|_{X^{0,\frac{1}{2}-\delta}},
\end{aligned}$$

with $(\frac{1}{2}-\varepsilon), (\frac{1}{2}-\delta) \geq \frac{3}{8} \implies \varepsilon, \delta \leq \frac{1}{8}$. □

Combining the linear estimates and Lemma 3.6, we can complete the contraction mapping argument.

Theorem 3.7. (NLS) *on* $\mathbb{T}$*, with cubic nonlinearity* ($p = 3$) *is locally well-posed in* $L^2_x(\mathbb{T})$.

Proof. By Duhamel formulation, define the solution map as follows

$$\Gamma_{u_0}(u) := \eta(t)S(t)u_0 \pm i\eta(t/T)\int_0^t S(t-t')\left(|u|^2u\right)(t')\,dt',$$

with $u_0 \in L^2_x(\mathbb{T})$, $\eta \in \mathcal{S}_t(\mathbb{R})$ a smooth cutoff and $T \in (0,1)$. Let $0 < \varepsilon \ll 1$ and

$$B_R := \left\{u \in X^{0,\frac{1}{2}+\varepsilon} : \ \|u\|_{X^{0,\frac{1}{2}+\varepsilon}} < R\right\}$$

a ball of radius $R > 0$, to be determined later. We want to conduct the contraction mapping argument within B_R. We start by showing that $\Gamma_{u_0} : B_R \to B_R$, for fixed $u_0 \in L^2_x(\mathbb{T})$.

$$\begin{aligned}
\|\Gamma_{u_0}(u)\|_{X^{0,\frac{1}{2}+\varepsilon}} &\leq \|\eta(t)S(t)u_0\|_{X^{0,\frac{1}{2}+\varepsilon}} + \left\|\int_0^t \eta(t/T)S(t-t')\left(|u|^2u\right)(t')\,dt'\right\|_{X^{0,\frac{1}{2}+\varepsilon}} \\
&\leq c_1\|u_0\|_{L^2_x} + c_2\left\|\eta(t/T)|u|^2u\right\|_{X^{0,-\frac{1}{2}+\varepsilon}} \\
&\leq c_1\|u_0\|_{L^2_x} + c_2c_3T^{\varepsilon}\left\||u|^2u\right\|_{X^{0,-\frac{1}{2}+2\varepsilon}} \\
&\leq c_1\|u_0\|_{L^2_x} + c_2c_3c_4T^{\varepsilon}\|u\|^3_{X^{0,\frac{3}{8}}},
\end{aligned}$$

using (2.4), (2.5), (2.6) and (3.6). Since $X^{0,\frac{9}{16}} \subset X^{0,\frac{3}{8}}$, choosing $\varepsilon = \frac{1}{16}$, the previous inequality becomes

$$\|\Gamma_{u_0}(u)\|_{X^{0,\frac{9}{16}}} \leq \tilde{c_1}\|u_0\|_{L^2_x} + \tilde{c_1}T^{\frac{1}{16}}\|u\|^3_{X^{0,\frac{9}{16}}},$$

with $\tilde{c_1} = \max\{c_1, c_2c_3c_4\}$. Choosing $R = 2\tilde{c_1}\|u_0\|_{L^2_x}$

$$\|\Gamma_{u_0}(u)\|_{X^{0,\frac{9}{16}}} \leq \frac{R}{2} + \tilde{c_1}T^{\frac{1}{16}}R^3 < R,$$

by choosing T such that $\tilde{c_1}T^{\frac{1}{16}}R^3 < \frac{R}{2}$.
Now, we want to show that Γ_{u_0} is a contraction for fixed $u_0 \in L^2_x(\mathbb{T})$. Let $u, v \in B_R$,

$$\begin{aligned}
I := \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X^{0,\frac{9}{16}}} &= \left\|\eta(t/T)\int_0^t S(t-t')\left(|u|^2u - |v|^2v\right)(t')\,dt'\right\|_{X^{0,\frac{9}{16}}} \\
&\leq c_2\left\|\eta(t/T)\left(|u|^2u - |v|^2v\right)\right\|_{X^{0,-\frac{7}{6}}} \\
&\leq c_2c_3T^{\frac{1}{16}}\left\||u|^2u - |v|^2v\right\|_{X^{0,-\frac{6}{16}}}
\end{aligned}$$

using (2.5) and (2.6). Writing $|u|^2u - |v|^2v$ as a telescoping sum, using (3.6) and the fact that $u, v \in B_R$,

$$\begin{aligned}
I &\leq c_2c_3T^{\frac{1}{16}}\left\|(u-v)u\bar{u} + (u-v)v\bar{u} + (\bar{u}-\bar{v})v^2\right\|_{X^{0,-\frac{6}{16}}} \\
&\leq c_2c_3T^{\frac{1}{16}}\|(u-v)u\bar{u}\|_{X^{0,-\frac{6}{16}}} + \|(u-v)v\bar{u}\|_{X^{0,-\frac{6}{16}}} + \left\|(\bar{u}-\bar{v})v^2\right\|_{X^{0,-\frac{6}{16}}} \\
&\leq c_2c_3c_4T^{\frac{1}{16}}\|u-v\|_{X^{0,\frac{3}{8}}}\left(\|u\|^2_{X^{0,\frac{3}{8}}} + \|v\|_{X^{0,\frac{3}{8}}}\|u\|_{X^{0,\frac{3}{8}}} + \|v\|^2_{X^{0,\frac{3}{8}}}\right) \\
&\leq 2c_2c_3c_4T^{\frac{1}{16}}\|u-v\|_{X^{0,\frac{9}{16}}}\left(\|u\|^2_{X^{0,3/8}} + \|v\|^2_{X^{0,3/8}}\right) \\
&\leq \tilde{c_2}T^{\frac{1}{16}}R^2\|u-v\|_{X^{0,\frac{9}{16}}},
\end{aligned}$$

with $\tilde{c_2} = 2c_2c_3c_4$. Choosing T such that $\tilde{c_2}T^{\frac{1}{16}}R^2 < \frac{1}{2}$ gives that Γ_{u_0} is a contraction on B_R. Using Banach's Fixed Point Theorem, we conclude that there exists a unique solution $u : [0,T] \times \mathbb{T} \to \mathbb{C}$ to the equation, in B_R.
It only remains to show that Γ_{u_0} is continuous with respect to the initial data u_0. Let $u_0, v_0 \in L^2_x(\mathbb{T})$, and u, v the respective solutions. Repeating the analysis for the integral part, as a consequence of the conditions imposed on T,

$$\begin{aligned}
\|u-v\|_{X^{0,\frac{9}{16}}} &= \|\Gamma_{u_0}(u) - \Gamma_{v_0}(v)\|_{X^{0,\frac{9}{16}}} \\
&\leq \|\eta(t)S(t)\,(u_0 - v_0)\|_{X^{0,\frac{9}{16}}} + \left\|\eta(t/T)\int_0^t S(t-t')\left(|u|^2u - |v|^2v\right)(t')\,dt'\right\|_{X^{0,\frac{9}{16}}} \\
&\leq c_1\|u_0 - v_0\|_{L^2} + c_2\left\|\eta(t/T)\left(|u|^2u - |v|^2v\right)\right\|_{X^{0,-\frac{7}{16}}} \\
&\leq c_1\|u_0 - v_0\|_{L^2} + \frac{1}{2}\|u-v\|_{X^{0,\frac{9}{16}}}\,.
\end{aligned}$$

Then, since $\frac{9}{16} > \frac{1}{2}$, we can use the embedding $X^{0,\frac{9}{16}} \subset C_T L^2_x$, to obtain

$$\|u-v\|_{C_T L^2_x} \leq \|u-v\|_{X^{0,\frac{9}{16}}} \leq 2c_1\|u_0 - v_0\|_{L^2_x},$$

which is sufficient to show that Γ_{u_0} is locally uniformly continuous. □

3.3 Quintic NLS

In this section, we focus on 1-d quintic (NLS). As stated before, for the Strichartz estimates available for this case, L^6, the Lebesgue norm is controlled by a $X^{0+,\frac{1}{2}+}$-norm, thus one expects LWP in H^{0+}, but not in L^2.

It remains to show the following quintilinear estimate, which corresponds to Lemma 2.5 in [3].

Theorem 3.8 (Quintilinear Estimate in $\mathbb{T}$)**.** *Let $s > 0$ and $\eta \in \mathcal{S}_t$ a smooth time cutoff. There exist $0 < \rho, \rho' \ll 1$ such that*

$$\left\|\eta(t)\prod_{j=1}^{5} u_j\right\|_{X^{s,-\frac{1}{2}+\rho}} \lesssim \prod_{j=1}^{5}\|u_j\|_{X^{s,\frac{1}{2}-\rho'}}. \tag{3.7}$$

Proof. To simplify the proof consider $u_j := \eta^{1/6}(t)u_j$ and $v = \eta^{1/6}(t)v$, $j = 1, \dots, 5$, so that we can apply estimates that require time cutoffs without carrying them in the proof. Note that using duality, Parseval's identity and explicitly writing the convolution, gives

$$\begin{aligned}
&\left\|\prod_{j=1}^{5} u_j\right\|_{X^{s,-\frac{1}{2}+\rho}} \\
&\quad= \sup_{\|v\|_{X^{0,\frac{1}{2}-\rho}}\leq 1}\left|\int_{\mathbb{R}}\int_{\mathbb{T}}\langle\nabla\rangle^s\left(\prod_{j=1}^{5} u_j\right)\bar{v}\,dxdt\right| \\
&\quad= \sup\left|\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}}\langle n\rangle^s\mathcal{F}\left(\prod_{j=1}^{5} u_j\right)(\tau,n)\,\widehat{v}(-\tau,-n)\,d\tau\right| \\
&\quad= \sup\left|\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}}\int_{\substack{\tau=\tau_1-\tau_2\\+\tau_3-\tau_4+\tau_5}}\sum_{\substack{n=n_1-n_2\\+n_3-n_4+n_5}}\langle n\rangle^s\left\{\prod_{j=1}^{5}\widehat{u_j}(\tau_j,n_j)\right\}\widehat{v}(-\tau,-n)\,d\tau_1\dots d\tau_4 d\tau\right| =: I.
\end{aligned}$$

Considering a Littlewood-Paley decomposition in space with dyadic frequencies,

$$v = \sum_{N \text{ dyadic}} P_N v, \quad u_j = \sum_{N_j \text{ dyadic}} P_{N_j} u_j,$$

for $j = 1, \dots, 5$, and substituting into the previous equation, denoting $*$ and $**$ as the sets of integration for the convolution sum and integral, respectively,

$$I \lesssim \sum_N \sum_{N_1} \dots \sum_{N_5} \sup \left| \int_{\mathbb{R}} \sum_n \int_* \sum_{**} \langle n \rangle^s \left\{ \prod_{j=1}^5 \widehat{P_{N_j} u_j}(\tau_j, n_j) \right\} \widehat{P_N v}(-\tau, -n) \, d\tau_1 \dots d\tau_4 d\tau \right|.$$

Now we need to consider different cases before estimating, focusing. As a consequence of symmetry, we can assume $N_1 \geq N_2 \geq \dots N_5$. First, we look at the case $N \lesssim 1$, and for $1 \lesssim N$, consider two options: $N_1 \sim N_2$ or $N_1 \sim N$.

<u>Case 1:</u> $N \lesssim 1$
Note that $\langle n \rangle^s \sim \langle N \rangle^s \lesssim 1$. Thus, from the previous equation, we get

$$I \lesssim \sum_{N \lesssim 1} \sum_{N_1, \dots, N_5} \sup \left| \int_{\mathbb{R}} \sum_n \int_* \sum_{**} \left\{ \prod_{j=1}^5 \widehat{P_{N_j} u_j}(\tau_j, n_j) \right\} \widehat{P_N v}(-\tau, -n) \, d\tau_1 \dots d\tau_4 d\tau \right| =: II.$$

Using Plancherel's identity and Hölder's inequality

$$\begin{aligned} II &= \sum_{N \leq 1} \sum_{N_1, \dots, N_5} \sup \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \left(\prod_{j=1}^5 P_{N_j} u_j \right) \overline{P_N v} \, dx \, dt \right| \\ &\lesssim \sum_{N \leq 1} \sum_{N_1, \dots, N_5} \sup \left\{ \prod_{j=1}^5 \| P_{N_j} u_j \|_{L^6_{t,x}} \right\} \| P_N v \|_{L^6_{t,x}}. \end{aligned}$$

For each $j = 1, \dots, 5$, if $N_j \lesssim 1$, then $N_j^s \lesssim 1$, for $s \geq 0$, and using the interpolated estimate (3.5), for some small $0 < \beta \ll 1$

$$\begin{aligned} \sum_{N_j \lesssim 1} \| P_{N_j} u_j \|_{L^6_{t,x}} &\lesssim \sum_{N_j \lesssim 1} N_j^\beta N_j^s \| P_{N_j} u_j \|_{X^{0, \frac{1}{2} - \beta}} \\ &\leq \left(\sum_{N_j \lesssim 1} N_j^{2\beta} \right)^{1/2} \left(\sum_{N_j \lesssim 1} N_j^{2s} \| P_{N_j} u_j \|^2_{X^{0, \frac{1}{2} - \beta}} \right)^{1/2} \\ &\lesssim_\beta \| u_j \|_{X^{s, \frac{1}{2} - \beta}}, \end{aligned}$$

using Cauchy-Schwarz inequality and the fact that N_j are dyadic.

Otherwise, if $N_j \gtrsim 1$, $N_j^{-s} \lesssim 1$, for $s \geq 0$, and

$$\begin{aligned}
\sum_{N_j \gtrsim 1} \|P_{N_j} u_j\|_{L^6_{t,x}} &\lesssim \sum_{N_j \gtrsim 1} N_j^{-s} \|\langle \nabla \rangle^s P_{N_j} u_j\|_{L^6_{t,x}} \\
&\lesssim \sum_{N_j \gtrsim 1} N_j^{\beta - s} \|P_{N_j} u_j\|_{X^{s,\frac{1}{2}-\beta}} \\
&\leq \left(\sum_{N_j \gtrsim 1} N_j^{2(\beta - s)} \right)^{\frac{1}{2}} \left(\sum_{N_j \geq 1} \|P_{N_j} u_j\|^2_{X^{s,1/2-\beta}} \right)^{\frac{1}{2}} \\
&\lesssim \|u_j\|_{X^{s,\frac{1}{2}-\beta}},
\end{aligned}$$

for $\beta < s$. From now on, we do not distinguish cases $N_j \leq 1$ and $N_j \geq 1$ for the frequencies of u_j, $j = 1, \dots, 5$, as it is clear that it is possible to control the L^6-norm this way.
Focusing on v,

$$\begin{aligned}
\sum_{N \lesssim 1} N^{\beta} \|P_N v\|_{X^{0,\frac{1}{2}-\beta}} &\leq \left(\sum_{N \lesssim 1} N^{2\beta} \right)^{\frac{1}{2}} \left(\sum_{N \lesssim 1} \|P_N v\|^2_{X^{0,\frac{1}{2}-\beta}} \right)^{\frac{1}{2}} \\
&\lesssim_{\beta} \|v\|_{X^{0,\frac{1}{2}-\beta}} \\
&\leq 1,
\end{aligned}$$

given that $0 < \rho \leq \beta$. Combining the precious estimates gives

$$II \lesssim \prod_{j=1}^{5} \|u_j\|_{X^{0,\frac{1}{2}-\beta}}.$$

<u>Case 2:</u> $N \gtrsim 1$, $N_1 \sim N_2$
Note that $N \sim |n| \leq |n_1| + \dots + |n_5| \lesssim 5N_1 \sim 5N_2$, then $N^s \lesssim N_1^{\frac{s}{2}} N_2^{\frac{s}{2}}$. Assuming $N_5 \gtrsim 1$,

$$\begin{aligned}
&\sup \left| \int_{\mathbb{R}} \sum_n \int_* \sum_{**} N^s \left\{ \prod_{j=1}^{5} \widehat{P_{N_j} u_j} \right\} \widehat{P_N v} \, d\tau_1 \dots d\tau_4 d\tau \right| \\
&\lesssim \sup \left| \int_{\mathbb{R}} \sum_n \int_* \sum_{**} N_1^{\frac{s}{2}} N_2^{\frac{s}{2}} \left\{ \prod_{j=1}^{5} \widehat{P_{N_j} u_j} \right\} \widehat{P_N v} \, d\tau_1 \dots d\tau_4 d\tau \right|.
\end{aligned}$$

Considering a spatial derivative of order s for each u_j, adds a power N_j^{-s}, $j = 1, \dots, 5$,

$$\begin{aligned}
(RHS) &\lesssim \sup \left| \int_{\mathbb{R}} \sum_n \int_* \sum_{**} N_1^{\frac{s}{2}} N_2^{\frac{s}{2}} \left\{ \prod_{j=1}^{5} \langle n_j \rangle^s N_j^{-s} \widehat{P_{N_j} u_j} \right\} \widehat{P_N v} \, d\tau_1 \dots d\tau_4 d\tau \right| \qquad (3.8) \\
&\lesssim \sup \left\{ N_1^{-\frac{s}{2}} N_2^{-\frac{s}{2}} N_3^{-s} N_4^{-s} N_5^{-s} \left(\prod_{j=1}^{5} \|\langle \nabla \rangle^s P_{N_j} u_j\|_{L^6_{t,x}} \right) \|P_N v\|_{L^6_{t,x}} \right\} \\
&\lesssim \sup \left\{ (N_1 N_2)^{-\frac{s}{2}+\beta} (N_3 \dots N_5)^{-s+\beta} N^{\beta} \left(\prod_{j=1}^{5} \|P_{N_j} u_j\|_{X^{s,\frac{1}{2}-\beta}} \right) \|P_N v\|_{X^{0,\frac{1}{2}-\beta}} \right\}
\end{aligned}$$

using Hölder's inequality and (3.5). Note that $N \gtrsim 1$ so the power series N^β does not converge. Since $N \lesssim N_j$, $j = 1, 2$, then $N_1^{-\alpha} \lesssim N^{-\alpha}$, for $\alpha > 0$. Let $\alpha = \frac{1}{4}$, then

$$(RHS) \lesssim \sup \Big\{ (N_1 N_2)^{-\frac{s}{4}+\beta} (N_3 \dots N_5)^{-s+\beta} N^{-\frac{s}{2}+\beta} \left(\prod_{j=1}^{5} \|P_{N_j} u_j\|_{X^{s,\frac{1}{2}-\beta}} \right) \|P_N v\|_{X^{0,\frac{1}{2}-\beta}} \Big\},$$

considering the sums and applying Cauchy-Schwarz inequality to each one, gives convergent sums if $\beta < \frac{s}{4}$, since N, N_j are dyadic, thus

$$I \lesssim \prod_{j=1}^{5} \|u_j\|_{X^{s,\frac{1}{2}-\beta}}.$$

However, it is possible that $N_j \lesssim 1$, for $j = 3, 4, 5$. In that case, we do not multiply the respective terms by $\langle n_j \rangle^s N^{-s}$ in (3.8), and it is sufficient to use (3.5), followed by Cauchy-Schwarz inequality, which gives

$$\begin{aligned} \sum_{N_j \le 1} N_j^\beta \|P_{N_j} u_j\|_{X^{0,\frac{1}{2}-\beta}} &\le \left(\sum_{N_j \le 1} N_j^{2\beta} \right)^{\frac{1}{2}} \left(\sum_{N_j \le 1} \|P_{N_j} u_j\|^2_{X^{0,\frac{1}{2}-\beta}} \right)^{\frac{1}{2}} \\ &\lesssim \|u_j\|_{X^{0,\frac{1}{2}-\beta}} \lesssim \|u_j\|_{X^{s,\frac{1}{2}-\beta}}, \end{aligned}$$

since $X^{s,\frac{1}{2}-\beta} \subset X^{0,\frac{1}{2}-\beta}$.

<u>Case 3:</u> $N \gtrsim 1$, $N_1 \gg N_2 \implies N_1 \sim N$
Let $\operatorname{supp} \widehat{P_{N_1} u_1} \subset I_1$, $\operatorname{supp} \widehat{P_N v} \subset I_2$, then $|I_1| \sim N_1 \sim N$, $|I_2| \sim N$. Since $N_1 \gg N_2$, we can decompose I_1 and I_2 in dyadic intervals of length N_2,

$$I_1 = \bigcup_l I_{1,l}, \quad I_2 = \bigcup_k I_{2,k}, \quad |I_{1,l}| \sim N_2, \quad |I_{2,k}| \sim N_2.$$

There are approximately N_1/N_2 intervals needed, therefore, the decompositions are finite and the following sums are finite

$$u_1 = \sum_l P_{I_{1,l}} u_1 =: \sum_l u_{1,l}, \quad v = \sum_k P_{I_{2,k}} v =: \sum_k v_k.$$

Note that if $n_1 \in I_1$, $n \in I_2$, more specifically $n_1 \in I_{1,l}$, $n \in I_{2,k}$ for some l, k, since

$$|n - n_1| = |n_2 - n_3 + n_4 - n_5| \le 4|n_2| \sim 4N_2,$$

then there exists a constant $c > 0$ such that $|l - k| \le cN_2 =: c_2$.
Using this result, we can substitute the sum over $N \gtrsim 1$ by a sum in k which satisfies

$|l-k| \leq c_2$,

$$
\begin{aligned}
I &\lesssim \sum_{\substack{N_1 \gg N_2, \\ N_4 \overset{\cdots}{\geq} N_5}} \sup \left| \int_{\mathbb{R}} \sum_n \int_* \sum_{**} N^s \left(\sum_{k,l} \widehat{u_{1,l}} \widehat{v_k} \right) \left\{ \prod_{j=2}^5 \widehat{P_{N_j} u_j} \right\} d\tau_1 \dots d\tau_4 d\tau \right| \\
&\lesssim \sum_{\substack{N_1 \gg N_2, \\ N_4 \overset{\cdots}{\geq} N_5}} \sum_{k,l} \sup \left| \int_{\mathbb{R}} \sum_n \int_* \sum_{**} N_1^s \widehat{u_{1,l}} \widehat{v_k} \left\{ \prod_{j=2}^5 \widehat{P_{N_j} u_j} \right\} d\tau_1 \dots d\tau_4 d\tau \right| \\
&\lesssim \sum_{\substack{N_1 \gg N_2, \\ N_4 \overset{\cdots}{\geq} N_5}} \sum_{k,l} \sup \left| \int_{\mathbb{R}} \sum_n \int_* \sum_{**} \langle n_1 \rangle^s \widehat{u_{1,l}} \widehat{v_k} \left\{ \prod_{j=2}^5 \langle n_j \rangle^s N_j^{-s} \widehat{P_{N_j} u_j} \right\} d\tau_1 \dots d\tau_4 d\tau \right| .
\end{aligned}
$$

Using Hölder's inequality and (3.5)

$$
\begin{aligned}
I &\lesssim \sum_{\substack{N_1 \gg N_2, \\ N_4 \overset{\cdots}{\geq} N_5}} \sum_{k,l} \sup \left(N_2^{-s} \dots N_5^{-s} \| \langle \nabla \rangle^s u_{1,l} \|_{L^6_{t,x}} \| v_k \|_{L^6_{t,x}} \prod_{j=2}^5 \| \langle \nabla \rangle^s P_{N_j} u_j \|_{L^6_{t,x}} \right) \\
&\lesssim \sum_{\substack{N_1 \gg N_2, \\ N_4 \overset{\cdots}{\geq} N_5}} \sum_{k,l} \sup \left(N_2^{\beta} N_2^{\beta} \left(N_2 \dots N_5 \right)^{-s+\beta} \| u_{1,l} \|_{X^{s,\frac{1}{2}-\beta}} \| v_k \|_{X^{0,\frac{1}{2}-\beta}} \prod_{j=2}^5 \| P_{N_j} u_j \|_{X^{s,\frac{1}{2}-\beta}} \right),
\end{aligned}
$$

since $u_{1,l}$ and v_k are projected onto intervals of size N_2. As before, assume that $N_5 \gtrsim 1$, otherwise treat $N_j \lesssim 1$ as in the previous case, $j = 2, 3, 4, 5$. Then, using Cauchy-Schwarz inequality, on the sums N_j, $j = 2, \dots, 5$, gives finite series and

$$
\begin{aligned}
I &\lesssim \sum_{N_1 \gg N_2} \sum_{k,l} \sup \left(\| u_{1,l} \|_{X^{s,\frac{1}{2}-\beta}} \| v_k \|_{X^{0,\frac{1}{2}-\beta}} \right) \prod_{j=2}^5 \left(\sum_{N_j \geq 1} \| P_{N_j} u_j \|^2_{X^{s,\frac{1}{2}-\beta}} \right)^{\frac{1}{2}} \\
&\lesssim \sup \sum_{N_1 \gg N_2} \sum_{k,l} \| u_{1,l} \|_{X^{s,\frac{1}{2}-\beta}} \| v_k \|_{X^{0,\frac{1}{2}-\beta}} \prod_{j=2}^5 \| u_j \|_{X^{s,\frac{1}{2}-\beta}} \\
&\lesssim \sup \left(\sum_{N_1} \sum_l \| u_{1,l} \|^2_{X^{s,\frac{1}{2}-\beta}} \right)^{\frac{1}{2}} \left(\sum_{N_1} \sum_k \| v_j \|^2_{X^{0,\frac{1}{2}-\beta}} \right)^{\frac{1}{2}} \prod_{j=2}^5 \| u_j \|_{X^{s,\frac{1}{2}-\beta}} \\
&\lesssim \prod_{j=1}^5 \| u_j \|_{X^{s,\frac{1}{2}-\beta}} .
\end{aligned}
$$

Combining the different cases, gives the intended result. □

Remark. Bourgain contributed with the idea of partitioning the biggest frequencies in intervals of the size of the small frequency, in order to gain summable powers of the small frequency, when applying L^6-Strichartz. This allowed to treat the most difficult case in the previous proof, case 3.

We can now combine the previous results to prove local well-posedness of quintic NLS in $H^{\varepsilon}(\mathbb{T})$, for $\varepsilon > 0$.

Theorem 3.9. (NLS) *on* $\mathbb{T}$*, with quintic nonlinearity* ($p = 5$)*, is locally well-posed in* $H^{\varepsilon}(\mathbb{T})$*,* $\varepsilon > 0$*.*

Proof. The proof follows the same ideas as in Theorem 3.7. The contraction mapping argument is run in $X^{\varepsilon,\frac{1}{2}+\rho}$, with $0 < \rho \ll 1$ small enough.
To show that $\Gamma_{u_0} : B_R \to B_R$, one applies linear estimates (2.4) and (2.5), followed by the quintilinear estimate (3.7) with $\rho' = \rho$, and lastly Time-Localisation estimate (2.6), gaining a power $T^{\frac{\rho}{2}}$.
To complete the contraction argument, use the same estimates and write $|u|^4u - |v|^4v$ as a telescoping sum,

$$|u|^4u - |v|^4v = (u - v)\left(|u|^4 + \overline{u}|u|^2\overline{v} + |u|^2|v|^2 + \overline{u}|v|^2\overline{v} + |v|^4\right).$$

Local uniform continuity follows from application of the same estimates. □

4 Korteweg-de Vries Type Equations

For Korteweg-de Vries type equations, we want to run the contraction mapping argument in $X^{s,b}$ with $b = \frac{1}{2}$. Consequently, we require a refinement of the space to guarantee the embedding into $C_t H_x^s$.

Definition 4.1. Let $s, b \in \mathbb{R}$. The space $Y^{s,b}(\mathbb{R} \times \mathbb{T})$, abbreviated $Y^{s,b}$, is defined as the closure of the Schwartz functions $\mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{T})$ under the *norm*

$$\|u\|_{Y^{s,b}} := \left\| \langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(\tau, n) \right\|_{\ell_n^2 L_\tau^1}.$$

Remark. The definition of the $Y^{s,b}$-spaces can be generalised to a polynomial h and to the d-dimensional torus.

It is easily seen that $Y^{s,b} \subset C_t H_x^s$ for $b \geq 0$. For simplicity, we introduce the following notation

$$Y^s := X^{s,\frac{1}{2}} \cap Y^{s,0}$$
$$Z^s := X^{s-\frac{1}{2}} \cap Y^{s,-1},$$

and we will conduct the contraction mapping argument in Y^s.

The linear estimates stated in Chapter 2 have analagous versions in Y^s. Refer to [4] for the proofs.

Proposition 4.2 (Homogeneous Linear Estimate)**.** *Let $u_0 \in H_x^s$ and $\eta \in \mathcal{S}_t$ a smooth time cutoff. Then,*

$$\|\eta(t) e^{-t\partial_x^3} u_0\|_{Y^s} \lesssim_{\eta,b} C \|u_0\|_{H_x^s}. \tag{4.1}$$

Proposition 4.3 (Inhomogeneous Linear Estimate)**.** *Let $s \in \mathbb{R}$, $b > \frac{1}{2}$, $F \in X^{s,b-1}$ and $\eta \in \mathcal{S}_t$ a smooth cutoff function. Then,*

$$\|\eta(t) \mathcal{I}(F)\|_{Y^s} \lesssim \|F\|_{Z^s}. \tag{4.2}$$

In 1993, Bourgain [2] showed local well-posedness of KdV in $L_x^2(\mathbb{T})$, but it was the following estimate, *bilinear estimate*, by Kenig-Ponce-Vega [6] which lead to improvement of LWP theory for KdV.

Proposition 4.4 (Bilinear Estimate, KPV '96 [6])**.** *Let $s \geq -\frac{1}{2}$, then*

$$\|\partial_x (\mathbb{P}(u)\mathbb{P}v)\|_{Z^s} \lesssim \|u\|_{X^{s,\frac{1}{2}}} \|v\|_{X^{s,\frac{1}{3}}} + \|u\|_{X^{s,\frac{1}{3}}} \|v\|_{X^{s,\frac{1}{2}}}, \tag{4.3}$$

with $\mathbb{P}$ projection onto the space of mean zero functions, $\mathbb{P}(u) = u - \int_{\mathbb{T}} u \, dx$.

Remark. One can state the bilinear estimate in a simpler way

$$\|\mathbb{P}(u)\partial_x v\|_{Z^s} \lesssim \|u\|_{X^{s,\frac{1}{2}}} \|v\|_{X^{s,\frac{1}{2}}},$$

but the $X^{s,\frac{1}{3}}$-norms will be important to apply the time localisation estimate. Note that $b = \frac{1}{2}$ in this estimate justifies running the contraction mapping argument in $X^{s,\frac{1}{2}}$.

Regarding modified and generalised KdV, the sharper results were shown by Kenig-Ponce-Vega [6] and Colliander-Keel-Staffilani-Takaoka-Tao [4, 5]. We state the main nonlinear estimate by CKSTT, which coupled with the bilinear estimate, gives control over the nonlinearity. The interested reader can find the proof in [5].

Proposition 4.5 (Nonlinear Estimate, CKSTT '04 [5])**.** *For any $s \geq \frac{1}{2}$,*

$$\left\| \prod_{j=1}^{p} u_j \right\|_{X^{s-1,\frac{1}{2}}} \lesssim \prod_{j=1}^{p} \|u_j\|_{Y^s}. \tag{4.4}$$

In this chapter we start by proving Strichartz estimates. Using such estimates, one can show the Bilinear Estimate (Proposition 4.4) and counter-examples asserting that the assumptions are necessary for the result. Consequently, it is possible to show local well-posedness of KdV in $H^{-\frac{1}{2}}(\mathbb{T})$, with a fixed mean.

For mKdV and gKdV, the bilinear estimate is not sufficient, thus we require Proposition 4.5 to complete the contraction mapping argument. We include an overview of the argument, without technical details.

The interested reader should refer to [2, 4, 5, 6] for more details.

4.1 Strichartz Estimates

The improvement in local well-posedness theory for NLS is partly due to Bourgain's sharp L^4-Strichartz estimates, as for KdV.

Proposition 4.6 (Bourgain '93, [2])**.** *Let u periodic in x and $\eta \in \mathcal{S}_t$ a smooth time cutoff, then*

$$\|\eta(t)u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0,\frac{1}{3}}}. \tag{4.5}$$

Proof. The proof is analogous to that of Proposition 3.3, with the change in the hypersurface defining $X^{s,b}$, and up to the estimation of (3.2), which becomes showing that

$$\sum_{n_1+n_2=n} \int_{\substack{\tau_1+\tau_2=\tau \\ \tau_1=n_1^3+\mathcal{O}(M) \\ \tau_2=n_2^3+\mathcal{O}(2^k M)}} 1 \, d\tau_1 \lesssim 2^{(\frac{2}{3}-2\varepsilon)k} M^{\frac{4}{3}}.$$

Since $\int 1 \, d\tau_1 = \mathcal{O}(M)$, because $\tau = n_1^3 + \mathcal{O}(M)$, for n_1 fixed, the previous reduces to showing

$$\sum_{\substack{n=n_1+n_2 \\ \tau=n_1^3+n_2^3+\mathcal{O}(2^k M)}} 1 \lesssim 2^{(\frac{2}{3}-2\varepsilon)k} M^{\frac{1}{3}}.$$

Since $n^3 - n_1^3 - n_2^3 = 3nn_1n_2$,

$$-3nn_1n_2 = \tau - n^3 + \mathcal{O}(2^k M) \implies 3n\left(n_1 - \frac{n}{2}\right)^2 = \tau - \frac{1}{4}n^3 + \mathcal{O}(2^k M).$$

If $|n| \gtrsim (2^k M)^{\frac{1}{3}}$, then

$$\left(n_1 - \frac{n}{2}\right)^2 = c_{k,n,\tau} + \mathcal{O}(2^k M)^{\frac{2}{3}},$$

showing that there are most $(2^k M)^{\frac{1}{3}}$ choices for n_1 and n_2.
When $|n| \ll (2^k M)^{\frac{1}{3}}$, by Minkowki's integral inequality and Cauchy-Schwarz inequality,

$$\begin{aligned}
\|u_M u_{2^k M}\|_{L^2_{t,x}} &= \Big\| \sum_{n=n_1+n_2} \int_{\mathbb{R}} \widehat{u_M}(\tau_1, n_1) \widehat{u_{2^k M}}(\tau - \tau_1, n_2)\, d\tau_1 \Big\|_{\ell^n_2 L62_\tau} \\
&\lesssim \Big\| \sum_{n=n_1+n_2} \int |\widehat{u_M}(\tau_1, n_1)| \|\widehat{u_{2^k M}}(\cdot, n_2)\|_{L^2_\tau}\, d\tau_1 \Big\|_{\ell^2_n} \\
&\lesssim \Big\| \sum_{n=n_1+n_2} \Big(\int\limits_{\tau_1 = n_1^3 + \mathcal{O}(M)} 1\, d\tau_1 \Big)^{\frac{1}{2}} \|\widehat{u_M}(\cdot, n_1)\|_{L^2_\tau} \|\widehat{u_{2^k M}}(\cdot, n_2)\|_{L^2_\tau} \Big\|_{\ell^2_n} \\
&\lesssim M^{\frac{1}{2}} \Big\| \sum_{n_1} \|\widehat{u_M}(\cdot, n_1)\|_{L^2_\tau} \|\widehat{u_{2^k M}}(\cdot, n - n_1)\|_{L^2_\tau} \Big\|_{\ell^2_n}.
\end{aligned}$$

Applying Cauchy-Schwarz inequality again, gives

$$\begin{aligned}
\|u_M u_{2^k M}\|_{L^2_{t,x}} &\lesssim M^{\frac{1}{2}} \Big(\sum_{|n| \ll (2^k M)^{\frac{1}{3}}} 1 \Big)^{\frac{1}{2}} \|\widehat{u_M}\|_{\ell^2_n L^2_\tau} \|\widehat{u_{2^k M}}\|_{\ell^2_n L^2_\tau} \\
&\lesssim 2^{\frac{k}{6}} M^{\frac{2}{3}},
\end{aligned}$$

which completes the proof. □

4.2 Bilinear Estimate

Before proving the bilinear estimate, we require an auxiliary result.

Lemma 4.7. *Fix $n \in \mathbb{Z} \setminus \{0\}$. For any $n_1, n_2 \in \mathbb{Z} \setminus \{0\}$, we have for all dyadic $M \geq 1$ that*

$$|\{\mu \in \mathbb{R} : \ |\mu| \sim M, \ \mu = -3nn_1n_1 + \mathcal{O}(\langle nn_1n_2 \rangle^\varepsilon\}| \leq M^{1-\delta},$$

for some $\delta > 0$.

Proof. Symmetry in n_1, n_2, allows us to assume $|n_1| \geq |n_2|$. Consider two cases: $|n| \geq |n_1|$ and $|n| \leq |n_1|$.
<u>Case 1:</u> $|n| \geq |n_1|$ Suppose $|\mu| \sim M$ and $|n| \sim N$, for M, N dyadic numbers. Then, since $|\mu| \lesssim |n|^3$ and $|n| \lesssim |\mu|$, for some $p \in [1, 3]$, $M \sim N^p$. In addition, since $\mu = -3nn_1n_2 + \mathcal{O}(\langle nn_1n_2 \rangle^\varepsilon)$, we must have $M \sim M^{\frac{1}{p}} |n_1n_2| \implies |n_1n_2| \sim M^{1-\frac{1}{p}}$. There are at most $M^{1-\frac{1}{p}}$ multiples of $M^{\frac{1}{p}}$ in the dyadic block $\{|\mu| \sim M\}$. Hence, the set of possible μ satisfying the previous relation must lie inside a union of $M^{1-\frac{1}{p}}$ intervals of size M^ε, each of which contains an integer multiple of n. Then,

$$|\{\mu \in \mathbb{R} : \ |\mu| \sim M, \ \mu = -3nn_1n_2 + \mathcal{O}(\langle nn_1n_2 \rangle^\varepsilon\}| < M^{1-\frac{1}{p}} M^\varepsilon \lesssim M^{\frac{3}{4}},$$

since $1 \leq p \leq 3$ and $\varepsilon \leq \frac{1}{12}$.
<u>Case 2:</u> $|n| \leq |n_1|$
We have that $|n_1| \lesssim |\mu|$ and $|\mu| \lesssim |n_1|^3$. If $|n_1| \sim N_1$, with N_1 dyadic, we must have $M \sim N_1^p$ for som $p \in [1, 3]$ and the same argument used in Case 1 holds. □

The proof of the bilinear estimate relies on the so called *denominator games*, introduced by Bourgain.

Proof of Proposition 4.4. Note that $\partial_x v = \mathbb{P}(\partial_x v)$ and $\|\partial_x v\|_{X^{s-1,\frac{1}{2}}} \lesssim \|v\|_{X^{s,\frac{1}{2}}}$. Thus, it suffices to show the following estimate

$$\|\partial_x \left(\mathbb{P}(u_1)\mathbb{P}(u_2)\right)\|_{Z^{s-1}} \lesssim \|u_1\|_{X^{s-1,\frac{1}{2}}} \|u_2\|_{X^{s-1,\frac{1}{2}}},$$

for $s \geq \frac{1}{2}$, stated in Proposition 4.4.
We start by showing the bound on the $X^{s,b}$-norm, $I := \|\partial_x \left(\mathbb{P}(u_1)\mathbb{P}(u_2)\right)\|_{X^{s-1,-\frac{1}{2}}}$,

$$\begin{aligned} I &= \left\| \langle n\rangle^{s-1} \langle \tau - n^3\rangle^{-\frac{1}{2}} |n| \widehat{u_1 u_2} \right\|_{\ell^2_n L^2_\tau} \\ &= \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \langle n\rangle^{s-1} |n| \langle \tau - n^3\rangle^{-\frac{1}{2}} \widehat{u_1}(\tau_1, n_1) \widehat{u_2}(\tau_2, n_2)\, d\tau_1 \right\|_{\ell^2 L^2_\tau}. \end{aligned}$$

Using the notation $\sigma := \langle \tau - n^3\rangle, \sigma_j := \langle \tau_j - n_j^3\rangle$, $j = 1, 2$, we have

$$I = \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\langle n\rangle^{s-1}|n|}{\langle n_1\rangle^{s-1}\langle n_2\rangle^{s-1}} \frac{1}{(\sigma\sigma_1\sigma_2)^{\frac{1}{2}}} \widehat{f_1}(\tau_1, n_2) \widehat{f_2}(\tau_2, n_2)\, d\tau_1 \right\|_{\ell^2 L^2_\tau},$$

with $\widehat{f_j} := \widehat{u_j}\langle n_j\rangle^{s-1}\sigma_j^{\frac{1}{2}}$. Thus, it suffices to show that $I \lesssim \|\widehat{f_1}\|_{\ell^2_n L^2_\tau} \|\widehat{f_2}\|_{\ell^2_n L^2_\tau}$.
We know that $n^3 - n_1^3 - n_2^3 = 3nn_1n_2$, so $\tau - n^3 = \tau_1 - n_1^3 + \tau_2 - n_2^3 - 3nn_1n_2$ and $|3nn_1n_2| \lesssim \max(\sigma, \sigma_1, \sigma_2)$. Since we are applying projections $\mathbb{P}$ to u_1, u_2 and the product, we can assume that $n, n_1, n_2 \neq 0$, as the associated Fourier coefficient for the 3 functions is equal to zero. Then, $|n| \sim \langle n\rangle, |n_j| \sim \langle n_j\rangle$, $j = 1, 2$, and using the fact that $\langle n\rangle^\alpha \lesssim \langle n_1\rangle^\alpha \langle n_2\rangle^\alpha$, for any $\alpha \geq 0$, assuming $s \geq \frac{1}{2}$

$$\frac{\langle n\rangle^{s-1}|n|}{\langle n_1\rangle^{s-1}\langle n_2\rangle^{s-1}} \lesssim \frac{\langle n_1\rangle^{s-\frac{1}{2}}\langle n_2\rangle^{s-\frac{1}{2}}|n|^{\frac{1}{2}}}{\langle n_1\rangle^{s-1}\langle n_2\rangle^{s-1}} \sim |nn_1n_2|^{\frac{1}{2}} \lesssim \max(\sigma, \sigma_1, \sigma_2)^{\frac{1}{2}}.$$

Substituting into the equation,

$$I \lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\max(\sigma, \sigma_1, \sigma_2)^{\frac{1}{2}}}{(\sigma\sigma_1\sigma_2)^{\frac{1}{2}}} \widehat{f_1}(\tau_1, n_2) \widehat{f_2}(\tau_2, n_2)\, d\tau_1 \right\|_{\ell^2 L^2_\tau}.$$

We must consider different cases, depending on the value of $\max(\sigma, \sigma_1, \sigma_2)$.

<u>Case 1:</u> $\sigma = \max(\sigma, \sigma_1, \sigma_2)$
Substituting the value of $\max(\sigma, \sigma_1, \sigma_2)$ gives

$$I \lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\widehat{f_1}}{\sigma_1^{\frac{1}{2}}} \frac{\widehat{f_2}}{\sigma_2^{\frac{1}{2}}}\, d\tau_1 \right\|_{\ell^2_n L^2_\tau}.$$

Denoting $\frac{\widehat{f_j}}{\sigma_j^{\frac{1}{2}}}$ by $\widehat{F_j}$, using Plancherel's identity, Hölder's inequality and L^4-Strichartz (4.5) gives

$$\begin{aligned} I &\lesssim \|F_1F_2\|_{L^2_{t,x}} \lesssim \|F_1\|_{L^4_{t,x}}\|F_2\|_{L^4_{t,x}} \lesssim \|F_1\|_{X^{0,\frac{1}{3}}}\|F_2\|_{X^{0,\frac{1}{3}}} \\ &= \|f_1\|_{X^{0,-\frac{1}{6}}}\|f_2\|_{X^{0,-\frac{1}{6}}} \\ &= \|u_1\|_{X^{s-1,\frac{1}{3}}}\|u_2\|_{X^{s-1,\frac{1}{3}}}. \end{aligned}$$

<u>Case 2:</u> $\sigma_1 = \max(\sigma, \sigma_1, \sigma_2)$
As a consequence of symmetry, this case is equivalent to $\sigma_2 = \max(\sigma, \sigma_1, \sigma_2)$. Substituting the maximum value in the expression gives

$$I \lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\widehat{f_1}}{\sigma^{\frac{1}{2}}} \frac{\widehat{f_2}}{\sigma_2^{\frac{1}{2}}}\, d\tau_1 \right\|_{\ell^2_n L^2_\tau}.$$

Using duality,

$$I \lesssim \sup_{\|\widehat{f_3}\|_{\ell^2_n L^2_\tau}\leq 1} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{f_1} \frac{\widehat{f_2}}{\sigma_2^{\frac{1}{2}}} \frac{\widehat{f_3}}{\sigma^{\frac{1}{2}}}\, d\tau_1\, d\tau.$$

Denoting $\widehat{f_2}\sigma_2^{-1/2}$, $\widehat{f_3}\sigma^{-1/2}$ by $\widehat{F_2}, \widehat{F_3}$, respectively, using Parseval's identity, Hölder's inequality and L^4-Strichartz (4.5),

$$\begin{aligned} I &\lesssim \sup\langle f_1F_2, F_3\rangle_{L^2_{t,x}} \lesssim \sup \|f_1\|_{L^2_{t,x}}\|F_2\|_{L^4_{t,x}}\|F_3\|_{L^4_{t,x}} \lesssim \sup \|f_1\|_{L^2_{t,x}}\|F_2\|_{X^{0,\frac{1}{3}}}\|F_3\|_{X^{0,\frac{1}{3}}} \\ &= \sup \|u_1\|_{X^{s-1,\frac{1}{2}}}\|u_2\|_{X^{s-1,\frac{1}{3}}}\|f_3\|_{X^{0,-\frac{1}{6}}} \\ &\leq \|u_1\|_{X^{s-1,\frac{1}{2}}}\|u_2\|_{X^{s-1,\frac{1}{3}}}. \end{aligned}$$

It remains to show the estimate for the $Y^{s-1,-1}$-norm, $II := \|\partial_x(\mathbb{P}(u_1)\mathbb{P}(u_2))\|_{Y^{s-1,-1}}$, following the same ideas as before

$$II \lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\max(\sigma,\sigma_1,\sigma_2)}{\sigma\sigma_1^{\frac{1}{2}}\sigma_2^{\frac{1}{2}}} \widehat{f_1}\widehat{f_2}\, d\tau_1 \right\|_{\ell^2_n L^1_\tau}.$$

<u>Case 1:</u> $\sigma_1 = \max(\sigma, \sigma_1, \sigma_2)$
As a consequence of symmetry, it is equivalent to consider $\sigma_2 = \max(\sigma, \sigma_1, \sigma_2)$. Substituting the value of $\max(\sigma, \sigma_1, \sigma_2)$

$$II \lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \sigma^{-1}\widehat{f_1}\frac{\widehat{f_2}}{\sigma_2^{\frac{1}{2}}}\, d\tau_1 \right\|_{\ell^2_n L^1_\tau}.$$

Denoting $\widehat{f_2}\sigma_2^{-1/2}$ by $\widehat{F_2}$, using Cauchy-Schwarz inequality on L^1_τ-norm gives

$$\begin{aligned} II &\lesssim \left\| \sigma^{-\frac{1}{2}-\varepsilon} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \sigma^{-\frac{1}{2}+\varepsilon} \widehat{f_1}\widehat{F_2}\, d\tau_1 \right\|_{\ell^2_n L^1_\tau} \\ &\leq \left\| \left(\int_{\mathbb{R}} \sigma^{-1-2\varepsilon}\, d\tau \right)^{\frac{1}{2}} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \sigma^{-\frac{1}{2}+\varepsilon} \widehat{f_1}\widehat{F_2}\, d\tau_1 \right\|_{\ell^2_n L^2_\tau}, \end{aligned}$$

for any $0 < \varepsilon$ the integral in τ is finite. Using duality, Parseval's identity, Hölder's inequality and L^4-Strichartz estimate (4.5)

$$\begin{aligned} II &\lesssim \sup_{\|\widehat{f_3}\|_{\ell^2_n L^2_\tau} \leq 1} \langle \widehat{f_1} \star \widehat{F_2}, \underbrace{\sigma^{-\frac{1}{2}+\varepsilon}\widehat{f_3}}_{=:\widehat{F_3}} \rangle_{\ell^2_n L^2_\tau} \\ &\lesssim \sup \langle f_1 F_2, F_3 \rangle_{L^2_{t,x}} \lesssim \sup \|f_1\|_{L^2_{t,x}} \|F_2\|_{L^4_{t,x}} \|F_3\|_{L^4_{t,x}} \lesssim \sup \|f_1\|_{L^2_{t,x}} \|F_2\|_{X^{0,\frac{1}{3}}} \|F^3\|_{X^{0,\frac{1}{3}}} \\ &= \sup \|u_1\|_{X^{s-1,\frac{1}{2}}} \|u_2\|_{X^{s-1,\frac{1}{3}}} \|f_3\|_{X^{0,-\frac{1}{6}+\varepsilon}} \\ &\leq \|u_1\|_{X^{s-1,\frac{1}{2}}} \|u_2\|_{X^{s-1,\frac{1}{3}}}, \end{aligned}$$

by choosing $0 < \varepsilon \leq \frac{1}{6}$.

<u>Case 2:</u> $\sigma = \max(\sigma, \sigma_1, \sigma_2)$
Substituting the value of $\max(\sigma, \sigma_1, \sigma_2)$ gives

$$II \lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{1}{(\sigma\sigma_1\sigma_2)^{\frac{1}{2}}} \widehat{f_1}\widehat{f_2}\, d\tau_1 \right\|_{\ell^2_n L^1_\tau}.$$

We require an extra assumption on the size of σ_1 in order to gain a small power of σ.
<u>Case 2.1:</u> $\sigma_1 \gtrsim |nn_1n_2|^\varepsilon$, for some $\varepsilon > 0$

With the extra assumption

$$\begin{aligned} II &\lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{|nn_1n_2|^{\frac{1}{2}}}{\sigma(\sigma_1\sigma_2)^{\frac{1}{2}}} \widehat{f_1}\widehat{f_2}\, d\tau_1 \right\|_{\ell^2_n L^1_\tau} \\ &\lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \sigma_1^\varepsilon \frac{\max(\sigma, \sigma_1, \sigma_2)^{\frac{1}{2}-\varepsilon}}{\sigma(\sigma_1\sigma_2)^{\frac{1}{2}}} \widehat{f_1}\widehat{f_2}\, d\tau_1 \right\|_{\ell^2_n L^1_\tau} \\ &\lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{1}{\sigma^{\frac{1}{2}+\varepsilon}\sigma_1^{\frac{1}{2}-\varepsilon}\sigma_2^{\frac{1}{2}}} \widehat{f_1}\widehat{f_2}\, d\tau_1 \right\|_{\ell^2_n L^1_\tau}. \end{aligned}$$

Using Cauchy-Schwarz inequality on the L^1_τ-norm, Hölder's inequality and Strichartz estimates, gives

$$\begin{aligned} II &\lesssim \left\| \left(\int_{\mathbb{R}} \sigma^{-1-2\varepsilon}\, d\tau \right)^{\frac{1}{2}} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \underbrace{\frac{\widehat{f_1}}{\sigma_1^{\frac{1}{2}-\varepsilon}}}_{=:\widehat{F_1}} \underbrace{\frac{\widehat{f_2}}{\sigma_2^{\frac{1}{2}}}}_{=:\widehat{F_2}}\, d\tau_1 \right\|_{\ell^2_n L^2_\tau} \\ &\lesssim \|F_1\|_{L^4_{t,x}} \|F_2\|_{L^4_{t,x}} \lesssim \|F_1\|_{X^{0,\frac{1}{3}}} \|F_2\|_{X^{0,\frac{1}{3}}} \\ &= \|u_1\|_{X^{s-1,\frac{1}{2}+\varepsilon}} \|u_2\|_{X^{s-1,\frac{1}{3}}}. \end{aligned}$$

<u>Case 2.2:</u> $\sigma_1, \sigma_2 \lesssim |nn_1n_2|^\varepsilon$, for some $\varepsilon > 0$
Note that $\tau - n^3 = (\tau_1 - n_1^3) + (\tau_2 - n_2^3) - 3nn_1n_2 = -3nn_1n_2 + \mathcal{O}(\langle nn_1n_2\rangle^\varepsilon)$. Then,

$$II \lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \sigma^{-\frac{1}{2}} \frac{\widehat{f_1}}{\sigma_1^{\frac{1}{2}}} \frac{\widehat{f_2}}{\sigma_2^{\frac{1}{2}}}\, d\tau_1 \mathbb{1}_{\Omega(n)}(\tau - n^3) \right\|_{\ell^2_n L^1_\tau},$$

with $\Omega(n) = \{\xi \in \mathbb{R} :\ \xi = -3nn_1n_2 + \mathcal{O}(\langle nn_1n_2\rangle^\varepsilon),\ \forall n_1, n_2 \in \mathbb{Z},\ n = n_1, n_2\}$
Using Cauchy-Schwarz inequality on the L^1_τ-norm, with $\widehat{f_j}\sigma_j^{-1/2}$ denoted by $\widehat{F_j}$, $j = 1, 2$, gives

$$II \left\| \lesssim \left(\int_{\mathbb{R}} \langle \tau - n^3\rangle^{-1} \mathbb{1}_{\Omega(n)}(\tau - n^3)\, d\tau \right)^{\frac{1}{2}} \widehat{F_1 F_2} \right\|_{\ell^2_n L^2_\tau}.$$

It remains to show that the integral is uniformly bounded with respect to n, since the remaining part can be bounded as before. Using Lemma 4.7, with $\mu = \tau - n^3$,

$$\begin{aligned} \int_{\mathbb{R}} \langle \mu\rangle^{-1} \mathbb{1}_{\Omega(n)}(\mu)\, d\mu &= \int_{|\mu|<1} \langle \mu\rangle^{-1}\, d\mu + \sum_{1<M \text{ dyadic}} \int_{|\mu|\sim M} \langle \mu\rangle^{-1} \mathbb{1}_{\Omega(n)}(\mu)\, d\mu \\ &\leq \log 1 + \sum_{1<M \text{ dyadic}} \frac{1}{M} |\Omega| \leq \sum_{1<M \text{ dyadic}} M^{-1} M^{1-\delta} \\ &= \sum_{1<M \text{ dyadic}} M^{-\delta} < \infty \end{aligned}$$

which concludes the proof. □

4.2.1 Counter-Examples for Bilinear Estimate

The conditions imposed on the bilinear estimate are necessary, as we intend to show. If we do not consider the projections $\mathbb{P}$ or $s < -\frac{1}{2}$ the estimate fails, and if the estimate holds, we must have $X^{s,\frac{1}{2}}$-norms on the right-hand side.

Focus on the simpler version of the bilinear estimate, assuming u_1, u_2 have mean zero, so that we can omit projections $\mathbb{P}$,

$$\|\partial_x(u_1u_2)\|_{X^{s,b-1}} \lesssim \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}}. \tag{BE}$$

Note that it is equivalent to prove

$$\left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s}{\sigma^{1-b}} \frac{f_1}{\sigma_1^b \langle n_1\rangle^s} \frac{f_2}{\sigma_2^b \langle n_2\rangle^s}\, d\tau_1 \right\|_{\ell^2_n L^2_\tau} \lesssim \|f_1\|_{\ell^2_n L^2_\tau} \|f_2\|_{\ell^2_n L^2_\tau}, \tag{BE'}$$

with $f_j := \langle n_1\rangle^s \sigma_j^s \widehat{u_j}$, $j = 1, 2$.

1. (BE) fails for any $s, b \in \mathbb{R}$ without mean zero condition. Let

$$\begin{aligned} f_1(\tau, n) &= \mathbb{1}_N(n)\mathbb{1}_{\leq 1}(\tau - n^3), \\ f_2(\tau, n) &= \mathbb{1}_0(n)\mathbb{1}_{\leq 1}(\tau - n^3). \end{aligned}$$

Then, $n = n_1 + n_2 = N$ and $\tau - n^3 = (\tau_1 - n_1^3) + (\tau_2 - n_2)^3 + 3nn_1n_2 = \mathcal{O}(1)$. Moreover, $\mathbb{1}_{\leq 1}(\tau_1 - N^3) \star \mathbb{1}_{\leq 1}(\tau_2) \geq \mathbb{1}_{\leq 1}(\tau - N^3)$. Hence,

$$\begin{aligned} &\left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s}{\sigma^{1-b}} \frac{\mathbb{1}_N(n_1)\mathbb{1}_{\leq 1}(\tau_1 - n_1^3)}{\sigma_1^b \langle n_1\rangle^s} \frac{\mathbb{1}_0(n_2)\mathbb{1}_{\leq 1}(\tau_2 - n_2^3)}{\sigma_2^b \langle n_2\rangle^s} \, d\tau_1 \right\|_{\ell_n^2 L_\tau^2} \\ &\sim \left\| N\mathbb{1}_{\leq 1}(\tau_1 - N^3) \star \mathbb{1}_{\leq 1}(\tau_2) \right\|_{L_\tau^2} \\ &\gtrsim N \| \mathbb{1}_{\leq 1}(\tau - N^3) \|_{L_\tau^2} \\ &\sim N. \end{aligned}$$

Looking at the right-hand side of the estimate,

$$\begin{aligned} \| \mathbb{1}_N(n_1)\mathbb{1}_{\leq 1}(\tau_1 - n_1^3) \|_{\ell_n^2 L_\tau^2} &\sim \| \mathbb{1}_{\leq 1}(\tau_1 - N^3) \|_{L_\tau^2} \sim 1, \\ \| \mathbb{1}_0(n_2)\mathbb{1}_{\leq 1}(\tau_2 - n_2^3) \|_{\ell_n^2 L_\tau^2} &\sim \| \mathbb{1}_{\leq 1}(\tau_2) \|_{L_\tau^2} \sim 1. \end{aligned}$$

Therefore, (BE') cannot hold for any $s, b \in \mathbb{R}$, and equivalently the bilinear estimate (BE) does not hold for any $s, b \in \mathbb{R}$ if the functions u_1, u_2 do not have mean zero.

2. If (BE) holds, then $b = \frac{1}{2}$. Let

$$\begin{aligned} f_1(\tau, n) &= \mathbb{1}_{N-1}(n)\mathbb{1}_{\leq 1}(\tau - n^3), \\ f_2(\tau, n) &= \mathbb{1}_1(n)\mathbb{1}_{\leq 1}(\tau - n^3). \end{aligned}$$

Then, $n = n_1 + n_2 = N$ and $\tau - n^3 = (\tau_1 - n_1^3) + (\tau_2 - n_2^3) + 3nn_1n_2 = \mathcal{O}(N^2)$. Looking at the left-hand side of (BE'), it is bounded from below by

$$\left\| \frac{N\langle N\rangle^s}{N^{2(1-b)}\langle N-1\rangle^s 2^s} \mathbb{1}_{\leq 1}(\tau_1 - (N-1)^3) \star \mathbb{1}_{\leq 1}(\tau_2 - 1) \right\|_{L_\tau^2} \gtrsim N^{2b-1},$$

using the same arguments as in the previous case. Similarly, looking at the right-hand side of (BE')

$$\| f_1 \|_{\ell_n^2 L_\tau^2} \| f_2 \|_{\ell_n^2 L_\tau^2} = \| \mathbb{1}_{\leq 1}(\tau_1 - (N-1)^3) \|_{L_{\tau_1}^2} \| \mathbb{1}_{\leq 1}(\tau - 1) \|_{L_{\tau_2}^2} \sim 1.$$

Therefore, $1 - 2b \leq 0 \implies b \leq \frac{1}{2}$. By duality, (BE') is equivalent to

$$\left| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s}{\langle \tau - n^3\rangle^{1-b}} \frac{f_1 f_2 f_3}{\langle n_1\rangle^s \langle n_2\rangle^2 \langle \tau_1 - n_1^3\rangle^b \langle \tau_2 - n_2^3\rangle^b} \, d\tau_1 \, d\tau \right| \lesssim \| f_1 \|_{\ell_n^2 L_\tau^2} \| f_2 \|_{\ell_n^2 L_\tau^2} \| f_3 \|_{\ell_n^2 L_\tau^2}. \quad \text{(BE'')}$$

Let

$$\begin{aligned} f_1(\tau_1, n_1) &= \mathbb{1}_{N-1}(n_1)\mathbb{1}_{\leq 1}(\tau_1 - n_1^3), \\ f_2(\tau_2, n_2) &= \mathbb{1}_1(n_2) \int \mathbb{1}_{\leq 1}(\tau - \tau_2 - (N-1)^3)\mathbb{1}_{\leq 1}(\tau - N^3)\, d\tau, \\ f_3(\tau, n) &= \mathbb{1}_N(n)\mathbb{1}_{\leq 1}(\tau - n^3). \end{aligned}$$

Then, $|\tau_2 - n_2^3| \sim N^2$. Looking at the left-hand side of (BE"), it is bounded from below by

$$\left| \int_{\tau=\tau_1+\tau_2} \frac{N\langle N\rangle^s}{\langle N-1\rangle^s 2^s N^{2b}} \mathbb{1}_{\leq 1}(\tau_1 - n_1^3)\mathbb{1}_{\leq 1}(\tau - n^3)\, d\tau_1\, d\tau \right| \gtrsim N^{1-2b}.$$

For the right-hand side of (BE"),

$$\|f_1\|_{\ell_n^2 L_\tau^2} \|f_2\|_{\ell_n^2 L_\tau^2} \|f_3\|_{\ell_n^2 L_\tau^2} \lesssim 1,$$

which implies $1 - 2b \leq 0 \implies b \geq \frac{1}{2}$. Consequently, $b = \frac{1}{2}$.

3. If (BE) holds, then $s \geq -\frac{1}{2}$. Let

$$\begin{aligned} f_1(\tau_1, n_1) &= \mathbb{1}_{N/2}(n_1)\mathbb{1}_{\leq 1}(\tau_1 - n_1^3), \\ f_2(\tau_2, n_2) &= \mathbb{1}_{N/2}(n_2)\mathbb{1}_{\leq 1}(\tau_2 - n_2^3). \end{aligned}$$

Then, $n = n_1 + n_2 = N$ and $\tau - n^3 \sim N^3$. Proceeding as before, we find that the left-hand side of (BE') is bounded from below by

$$\left\| \frac{N\langle N\rangle^s}{N^{\frac{3}{2}}} \int_{\tau=\tau_1+\tau_2} \frac{\mathbb{1}_{\leq 1}(\tau_1 - N^{\frac{3}{8}})\mathbb{1}_{\leq 1}(\tau_2 - N^{\frac{3}{8}})}{\langle N/2\rangle^{2s}}\, d\tau_1 \right\|_{L_\tau^2} \gtrsim N^{1-s} N^{\frac{3}{2}} = N^{-\frac{1}{2}-s}.$$

As before, the right-hand side of the estimate is similar to 1. Consequently, $-\frac{1}{2} - s \leq 0 \implies -\frac{1}{2} \leq s$.

4.3 Korteweg-de Vries Equation

We have the tools to prove local well-posedness of KdV. However, to apply the Bilinear Estimate (Proposition 4.4), we must have mean zero functions. Thus, consider the *gauge transformation* $u(t,x) \mapsto v(t,x) := u(t, x \mp ct)$, for some constant $c \in \mathbb{R}$. Then, v solves the following equation

$$\partial_t v + \partial_x^3 v = \pm(v - c)\partial_x v, \tag{4.6}$$

with initial data $v_0 = u_0$.

It is easily seen that if v is smooth we have conservation of mean, i.e., $\int_{\mathbb{T}} v(t,x)\, dx = \int_{\mathbb{T}} v_0(x)\, dx$, for all $t \in \mathbb{R}$. By a density argument, the conservation of mean can be extended to $v \in C_t H_x^s$. Therefore, we can fix $c = \int_{\mathbb{T}} v_0(x)\, dx$, so that in (4.6) the two factors in the nonlinearity have mean zero, $\left(v - \int_{\mathbb{T}} v\, dx\right)$ and $\partial_x v$.

By applying the gauge transformation with fixed $c \in \mathbb{R}$, we are prescribing a mean to the initial data. Therefore, we can prove the following result.

Theorem 4.8 (KPV '96, [6]). *The Korteweg-de Vries equation, (KdV) with $p = 1$ and prescribed mean, is locally well-posed in $H^s(\mathbb{T})$ for $s \geq -\frac{1}{2}$ and initial data u_0 with fixed mean.*

Remark. The LWP in $H^{-\frac{1}{2}}$ is a consequence of condition $s \geq -\frac{1}{2}$ in the bilinear estimate (Proposition 4.4). Similarly, the need to apply the gauge transformation restricts the result to initial data with fixed mean. Without fixing the mean, one can show failure of uniform continuity of the solution map.

Proof of Theorem 4.8. The proof follows the same ideas as in Theorem 3.7.
By Duhamel formulation, define the solution map as follows

$$\Gamma_{u_0}(u) := \eta(t)S(t)u_0 \pm \eta(t/T)\int_0^t S(t-t')\left((u - \int_{\mathbb{T}} u\,dx)\partial_x u\right)(t')\,dt',$$

with $\mathcal{F}\{S(t)f\}(n) = e^{itn^3}\widehat{f}(n)$, $u_0 \in H^s(\mathbb{T})$ with fixed norm, $\eta \in \mathcal{S}_t(\mathbb{R})$ a smooth cutoff and $T \in (0,1)$.
The contraction mapping argument is run in Y^s.
To show that $\Gamma_{u_0} : B_R \to B_R$, one applies linear estimates (4.1) and (4.2), followed by the bilinear estimate (4.3). Lastly, Time Localisation estimate (2.6) gives a power T^θ, for $0 < \theta < \frac{1}{6}$.
To complete the contraction mapping argument and prove local uniform continuity, use the same estimates, and proceed as in the proof for Theorem 3.7. □

4.4 Modified and Generalised KdV

In this section, we present the heuristics behind the contraction mapping argument for mKdV and gKdV, without technical details. For the interested reader, refer to [5].

As for KdV, the bilinear estimate requires a modification of the equation. Considering the gauge transfomation $u(t,x) \mapsto v(t,x) = u(t, x \mp ct)$ with $c = \int_{\mathbb{T}} u^p(t,x)\,dx$, which one can show to be conserved, gives

$$\partial_t v + \partial_x^3 v = \pm\left(v^p - \int_{\mathbb{T}} v^p\,dx\right)\partial_x v.$$

Now, we can apply the bilinear estimate, but is not sufficient to complete the contraction mapping argument. Thus, one needs multilinear estimate Proposition 4.5 to control the nonlinearity. Applying the bilinear estimate to the nonlinear term gives

$$\left\|\eta(t/T)\left(v^p - \int_{\mathbb{T}} v^p\,dx\right)\partial_x v\right\|_{Z^s} \lesssim \|v^p\|_{X^{s-1,\frac{1}{2}}}\|\eta(t/T)v\|_{X^{s-1,\frac{1}{3}}} + \|\eta(t/T)v^p\|_{X^{s-1,\frac{1}{3}}}\|v\|_{X^{s-1,\frac{1}{2}}},$$

one can gain a small power of T by using the Time Localisation estimate, $0 < \theta < \frac{1}{6}$,

$$\left\|\eta(t/T)\left(v^p - \int_{\mathbb{T}} v^p\,dx\right)\partial_x v\right\|_{Z^s} \lesssim T^\theta\|v^p\|_{X^{s-1,\frac{1}{2}}}\|v\|_{X^{s-1,\frac{1}{2}}}.$$

In order to control the norm of v^p, we require a nonlinear estimate, Proposition 4.5. However, this estimate requires $s \geq \frac{1}{2}$, which restricts the range of s for LWP. Using the detailed approach, one can prove the following result.

Theorem 4.9 (CKSTT '04 [5]). *Modified and Generalised KdV,* (KdV) *with $p \geq 2$, with prescribed L^p-norm, are locally well-posed in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$.*

5 Conclusion

The extended project focused on understanding the Fourier Restriction Norm Method, proposed by Bourgain in 1993, to prove local well-posedness of nonlinear Schrödinger and Korteweg-de Vries type equations, in the periodic setting.

The method uses harmonic analytic tools, through a contraction mapping argument in the $X^{s,b}$ spaces. Recall that the space-time Fourier Transform of the linear homogeneous solution is supported on a characteristic hypersurface. Consequently, as the method follows a perturbative approach, the $X^{s,b}$ spaces exploit the expected proximity of the nonlinear solution, after time localisation, through its norm that penalises functions which lie far from the hypersurface.

For the argument, one requires linear estimates, which are easily proven, and, most importantly, nonlinear estimates. For the latter, Strichartz estimates are extremely useful. However, unlike the real line setting, one cannot use the dispersion relation. Despite such difficulties, Bourgain in [2] improved existing Strichartz estimates, simply with Fourier analytic methods, which allowed for sharper nonlinear estimates.

For the nonlinear Schrödinger equation, Bourgain in [2] improved many of the known well-posedness results, with two examples mentioned in Chapter 3, cubic and quintic NLS on $\mathbb{T}$. The nonlinear estimates presented exemplify the classical techniques associated with the method.

For the Korteweg-de Vries type equations, Kenig, Ponce and Vega in [6] showed the so-called Bilinear Estimate. Such result was sufficient to close the argument for KdV, but not for higher order nonlinearities. Thus, for modified and generalised KdV, CKSTT in [5] showed additional nonlinear estimates. It is important to note the need to use the gauge transformation, which narrows down the results to certain families of inital data. Namely, for nonlinearity $u^p\partial_x u$ one needs to prescribe the L^p-norm of the initial data to have uniform continuity of the solution map.

In the future, the author intends to focus on ill-posedness results for the KdV type equations, and consider the Fourier Restriction Norm Method in the context of Fourier-Lebesgue spaces, for modified and generalised KdV in the periodic setting.

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