· C. Fefferman 1971

The multiplier problem for the ball.

T, defined by $\widehat{Tf}(\xi) = \chi_B(\xi)\hat{f}(\xi)$

where $B = B(0,1)$.

<u>Q</u>: Is T bounded on $L^p(\mathbb{R}^n)$.

i.e. $\chi_B \in \mathcal{M}_p(\mathbb{R}^n)$?

the set of L^p Fourier multipliers.

$$\mathcal{M}_p(\mathbb{R}^n) = \left\{ m \;\middle|\; \begin{array}{l} T_m : L^p \to L^p \\ \text{where } T_m(f) = (m\hat{f})^\vee, \quad f \in \mathcal{S} \end{array} \right\}$$

$$\mathcal{M}^{p,q}(\mathbb{R}^n) = \{ T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \text{ commuting with translations} \}$$

$\Updownarrow$

given by a convolution with a tempered distribution.

$$\Rightarrow \quad m \in \mathcal{M}_p(\mathbb{R}^n) \Longleftrightarrow T_m \in \mathcal{M}^{p,p}(\mathbb{R}^n)$$

Some basic facts:

1. $\mathcal{M}^{p,q} = \{0\}$ if $1 \le q < p \le \infty$.

2. $T \in \mathcal{M}^{p,q} \iff T \in \mathcal{M}^{q',p'}$

   In particular, $\mathcal{M}_p(\mathbb{R}^n) = \mathcal{M}_{p'}(\mathbb{R}^n)$

3. $T \in \mathcal{M}^{1,1} = \mathcal{M}^{\infty,\infty}$

   $\iff$ T is given by convolution with a finite ($\mathbb{C}$-valued) Borel measure μ. (and $\|T\| = \|\mu\|$ ← total variation)

4. $T \in \mathcal{M}^{2,2}$

   $\iff$ T is given by convolution with some $u \in \mathcal{S}'$ s.t. $\hat{u} \in L^\infty$ (and $\|T\| = \|\hat{u}\|_{L^\infty}$.)

5. $1 \le p \le q \le 2$,

$$\mathcal{M}_1 \subset \mathcal{M}_p \subset \mathcal{M}_q \subset \mathcal{M}_2 = L^\infty$$

6. $m(\xi, \eta) \in \mathcal{M}_p(\mathbb{R}^{n+m})$.

   Then, for a.e. $\xi \in \mathbb{R}^n$, the function $\eta \longmapsto m(\xi, \eta)$ is in $\mathcal{M}_p(\mathbb{R}^m)$ with

$$\|m(\xi, \cdot)\|_{\mathcal{M}_p(\mathbb{R}^m)} \le \|m\|_{\mathcal{M}_p(\mathbb{R}^{n+m})}$$

· In particular, if $\chi_B \notin \mathcal{M}_p(\mathbb{R}^2)$,
then, $\chi_B \notin \mathcal{M}_p(\mathbb{R}^n)$, $\forall\ n \geq 3$.

· <u>Known Results</u>

① T_{cube} is bounded on L^p, $1 < p < \infty$.

② Let $f(x) = \begin{cases} 1 & , |x| < 1/10 \\ 0 & , \text{otherwise.} \end{cases}$

$\Rightarrow$ Tf decays too slowly at ∞

and T can NOT be bounded on $L^p(\mathbb{R}^n)$

unless $\dfrac{2n}{n+1} < p < \dfrac{2n}{n-1}$

· <u>Disc Conjecture</u>: T is bounded on $L^p(\mathbb{R}^n)$

with $\dfrac{2n}{n+1} < p < \dfrac{2n}{n-1}$

3 Bochner-Riesz spherical summation operators

$$\widehat{T_\delta f}(\xi) = \left[\max\left(1-|\xi|^2, 0\right)\right]^\delta \hat{f}(x)$$

$\Rightarrow$ <u>Theorem</u> (Carleson & Sjölin)

T_δ is bounded on $L^p(\mathbb{R}^2)$ for $\delta > 0$

where $\dfrac{4}{3+2\delta} < p < \dfrac{4}{1-2\delta}$

In particular, for $\delta = 0+$, we have

$$\frac{4}{3} \le p \le 4$$

$$\frac{4}{3} = \frac{2n}{n+1}, \qquad 4 = \frac{2n}{n-1}$$

$\Leftarrow$ exact range for disc conjecture

- <u>More on 2</u>

<u>Recall</u>: for $g(x) = g_0(|x|)$ radial, we have

$$\hat{g}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty g_0(r)\, J_{\frac{n}{2}-1}(2\pi r|\xi|)\, r^{\frac{n}{2}}\, dr$$

(Also,

$$\widehat{d\sigma}(\xi) = \int_{S^{n-1}} e^{-2\pi i \xi\cdot\theta}\, d\theta = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|)$$

With $f(x) = \chi_{B(0,\frac{1}{10})}(x)$, we have

$$Tf(x) = \int_{|y|<\frac{1}{10}} \check{\chi}_B(x-y)\,dy$$

$=$ average of $\check{\chi}_B$ around x.

$$\circ\ \check{\chi}_B(x) = \frac{2\pi}{|x|^{\frac{n-2}{2}}} \int_0^1 J_{\frac{n}{2}-1}(2\pi r|x|)\, r^{\frac{n}{2}}\, dr$$

$(2\pi|x|)^{\frac{n}{2}+1}\ (2\pi|x|)^{-\frac{n}{2}-1}$

$\left(\text{use } \frac{d}{dz}\left(z^k J_k(z)\right) = z^k J_{k-1}(z) \text{ for } k > 1/2\right)$

$$= \frac{2\pi}{|x|^{\frac{n-2}{2}}} \frac{1}{(2\pi|x|)^{\frac{n}{2}+1}} z^{n/2} J_{n/2}(z) \Big|_0^{2\pi|x|}$$

<u>Recall</u> for $k > -1/2$

$$J_k(r) = \begin{cases} \dfrac{r^k}{2^k \Gamma(k+1)} + O(r^{k+1}) & \text{as } r \to 0 \\[2ex] \sqrt{\dfrac{2}{\pi r}} \cos\left(r - \dfrac{\pi k}{2} - \dfrac{\pi}{4}\right) + O(r^{-3/2}) & \text{as } r \to \infty \end{cases}$$

$$\Rightarrow \lim_{z \to 0} z^{\frac{n}{2}} J_{\frac{n}{2}}(z) = 0$$

$$\Rightarrow \check{\chi}_B(x) = \frac{J_{n/2}(2\pi|x|)}{|x|^{n/2}}$$

(Similarly, with $m_\delta(\xi) = [\max(1-|\xi|^2, 0)]^\delta$,

$$\check{m}_\delta(\xi) = \frac{\Gamma(\delta+1)}{\pi^\delta} \frac{J_{\frac{n}{2}+\delta}(2\pi|x|)}{|x|^{\frac{n}{2}+\delta}}$$

Hence, for $|x|$ large,

$|Tf(x)|$ = average of $\check{\chi}_B$ around x

$$\sim \frac{1}{|x|^{n/2}} \frac{1}{\sqrt{x}} = \frac{1}{|x|^{\frac{n+1}{2}}}$$

which is $\begin{cases} \text{in } L^p(\mathbb{R}^n) \text{ for } p > \frac{2n}{n+1} \\ \text{not in } L^p(\mathbb{R}^n) \text{ for } p \le \frac{2n}{n+1} \end{cases}$

By duality, $\left(p = \frac{2n}{n+1} \Rightarrow p' = \frac{2n}{n-1} \right)$

we have

$$p < \frac{2n}{n-1} \text{ if } T \text{ could be bounded.}$$

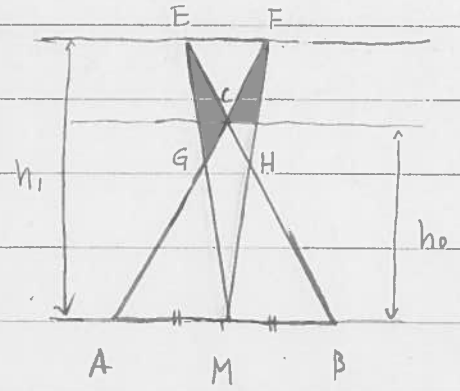
- It turned out that the disc conjecture is false.

<u>Theorem 1:</u> T is bounded only in L^2 $(n > 1)$

Note that it is enough to disprove L^p-boundedness of T <u>on $\mathbb{R}^2$</u> for <u>$p > 2$</u>.

- We begin with a certain geometric construction that at first sight has no apparent relationship with the multiplier problem for the ball

Given ΔABC, construct 2 triangles AMF, BME, called the sprouts of ABC.



$b = AB$. h_0 = height of ABC

M = Midpoint of AB.

$h_1 > h_0$

Let $Spr(ABC) = AMF \cup BME$

= the sprouted figure obtained from ABC

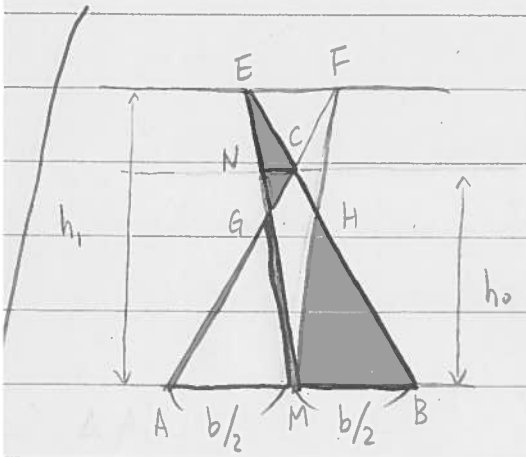
Clearly, $ABC \subset Spr(ABC)$

Also, let "arms of the sprouted figure" $= Spr(ABC) \setminus ABC$

- In this case, Spr (ABC) has two arms of equal area, EGC and FCH.

Moreover,

$$\text{Area}\left(\text{each arm of } Spr(ABC)\right) = \frac{b}{2}\,\frac{(h_1-h_0)^2}{2h_1-h_0}$$



- By similarities of Δ's,

$$\frac{NC}{b/2} = \frac{h_1-h_0}{h_1}$$

and

$$\frac{\text{height}(NGC)}{h_0} = \frac{NC}{NC+b/2}$$

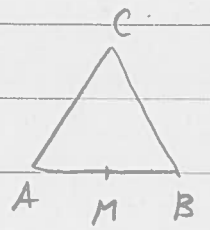
Then, height (EGC)

$$= h_0\,\frac{NC}{NC+b/2} + h_1 - h_0 = \dots = \frac{2h_1(h_1-h_0)}{2h_1-h_0}$$

and $\text{base}(EGC) = NC = \dfrac{h_1-h_0}{h_1}\cdot\dfrac{b}{2}$

$$\Rightarrow |\Delta EGC| = \frac{b}{2}\,\frac{(h_1-h_0)^2}{2h_1-h_0}$$

- Now, let $\Lambda = ABC$ be an isosceles triangle in $\mathbb{R}^2$ with $\begin{cases} \text{base } AB \text{ of length } b_0 = \varepsilon \\ \text{height } MC = h_0 = \varepsilon. \end{cases}$



Define the seq. of heights

$$h_1 = \left(1 + \frac{1}{2}\right)\varepsilon,$$

$$h_2 = \left(1 + \frac{1}{2} + \frac{1}{3}\right)\varepsilon$$

$$\vdots$$

$$h_j = \left(1 + \frac{1}{2} + \cdots + \frac{1}{j+1}\right)\varepsilon.$$

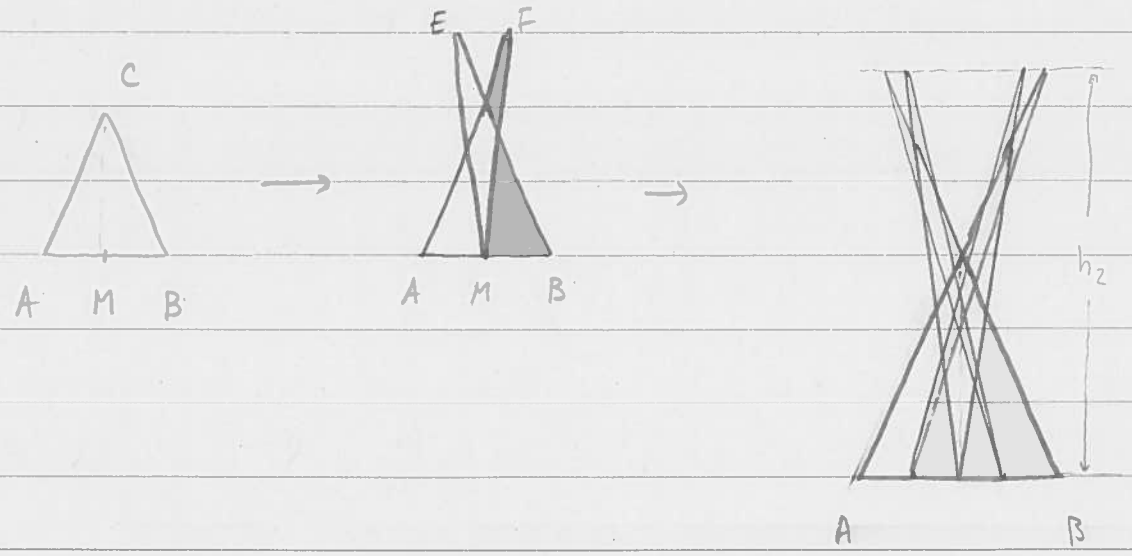
- Apply the sprouting procedure to Λ to obtain 2 sprouts $\Lambda_1 = AMF$, $\Lambda_2 = EMB$, each with height h_1 base length $b_0/2$.

- Now, apply the same procedure to Λ_1 and Λ_2 and obtain

Λ_{11}, Λ_{12} from Λ_1

Λ_{21}, Λ_{22} from Λ_2

4 sprouts with height h_2



- Continue this process obtaining, at the jth step, 2^j sprouts $\Lambda_{r_1 \cdots r_j}$, $r_1, \cdots, r_j \in \{1, 2\}$,

each with $\begin{cases} \text{base length} & b_j = 2^{-j} b_0 \\ \text{height} & h_j \end{cases}$

- Stop at the kth step and let

$$E(\varepsilon, k) = \bigcup_{r_j \in \{1,2\}} \Lambda_{r_1 \cdots r_k}$$

$\Rightarrow$ Area $(E(\varepsilon, k)) \leq |\Lambda|$ + Areas of arms of all the sprouted figures.

- At j^{th} step, each of 2^j arms has area

$$\frac{b_{j-1}}{2} \frac{(h_j - h_{j-1})^2}{2h_j - h_{j-1}}$$

$$\Rightarrow \left|E(\varepsilon, k)\right| \leq \frac{1}{2}\varepsilon^2 + \sum_{j=1}^{k} 2^j \frac{b_{j-1}}{2} \frac{(h_j - h_{j-1})^2}{2h_j - h_{j-1}}$$

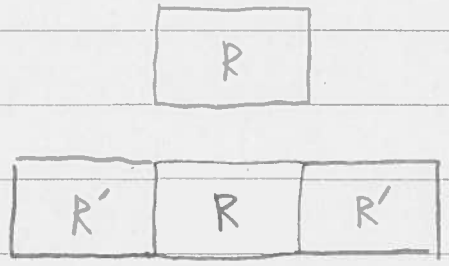
$$\leq \frac{1}{2}\varepsilon^2 + \sum_{j=1}^{k} 2^j \frac{2^{-(j-1)} b_0}{2} \frac{\varepsilon^2}{(j+1)^2 \varepsilon}$$

$$\left(\because \quad 2h_j - h_{j-1} \geq \varepsilon \quad \forall j \geq 1 \right)$$

$$\leq \frac{1}{2}\varepsilon^2 + \sum_{j=2}^{\infty} \frac{\varepsilon^2}{j^2} \leq \left(\frac{1}{2} + \frac{\pi^2}{6} - 1\right)\varepsilon^2 \leq \frac{3}{2}\varepsilon^2$$

$\Rightarrow$ For any k, $|E(\varepsilon, k)|$ can be made arbitrarily small if we take ε small.

- <u>Notation</u>: Given a rectangle $R \subset \mathbb{R}^2$, let R' be two copies of R adjacent to R along its short side s.t. $R \cup R'$ has the same width as R but 3 times its length



<u>Lemma 1</u>: $\delta > 0$. Then, $\exists$ measurable $E \subset \mathbb{R}^2$ and a finite collection of rectangles R_j in $\mathbb{R}^2$ s.t.

(1) The R_j's are pairwise disjoint

(2) $1/2 \le |E| \le 3/2$

(3) $|E| \le \delta \sum_j |R_j|$

(4) for all j, $|R_j' \cap E| \ge \frac{1}{12} |R_j|$

<u>Pf</u>) Let ABC be an isosceles triangle of height 1 with $A = (0,0)$ and $B = (1,0)$.

- Given $\delta > 0$, choose $k \in \mathbb{N}$ s.t. $k + 2 > e^{1/\delta}$.

For this k, set $E = E(1,k)$, the union of sprouted figures constructed earlier with $\varepsilon = 1$

Then, $1/2 \le E \le 3/2 \Rightarrow (2)$

- Recall each $\left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$ in $[0,1]$ is the base of exactly one sprouted triangle $A_j B_j C_j$

$(j = 0, 1, \cdots, 2^k - 1)$.

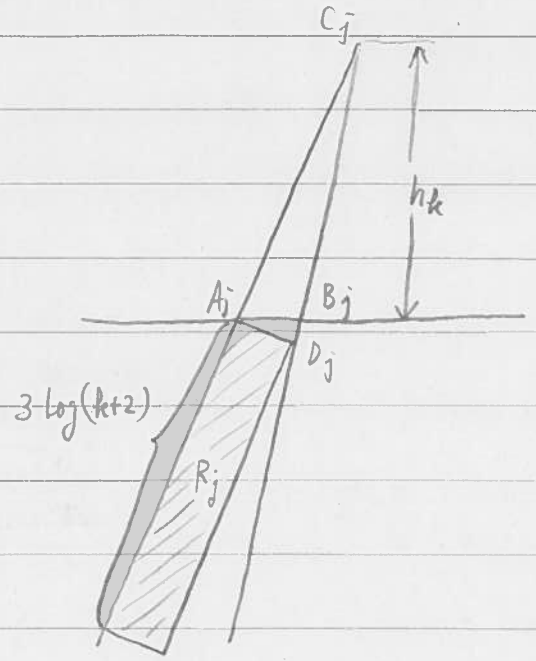
where $A_j = (j/2^k, 0)$, $B_j = (j+1/2^k, 0)$

and C_j = the other vertex of the sprouted triangle.

- Define R_j inside $\angle A_j C_j B_j$ as below

i.e. R_j is defined s.t. one vertex is either A_j or B_j

and the length of its long side is $3 \log(k+2)$.



- Note that $\max_j (A_j C_j, B_j C_j) = \dfrac{\sqrt{5}\, h_k}{2}$

($\because$ 

$\sqrt{1^2 + (\frac{1}{2})^2} = \frac{\sqrt{5}}{2}$

- WLOG, assume $A_jC_j \geq B_jC_j$.

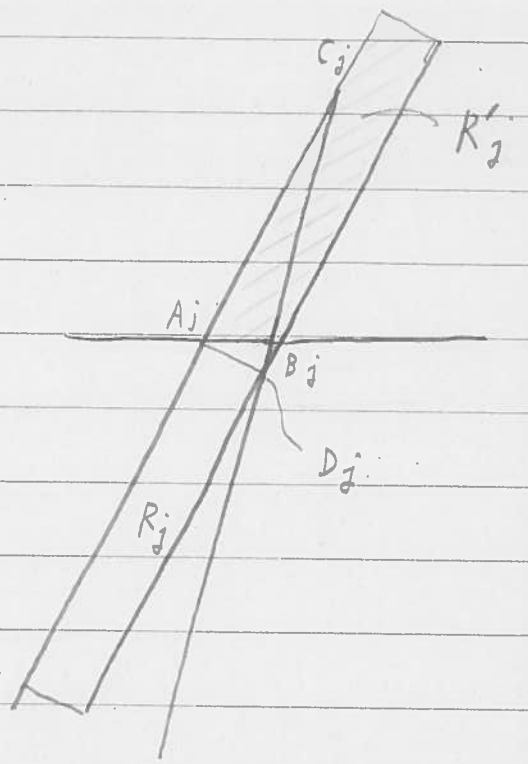
Then,

$$\frac{\sqrt{5}}{2} h_k < \frac{3}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k+1}\right)$$

$$< \frac{3}{2}\left(1 + \log(k+1)\right) < 3\log(k+2).$$

$$\left(\because\ k \geq 1 \text{ and } e < 3 \Rightarrow 1 < \log(k+2) \right)$$

Hence, R_j' contains $A_jB_jC_j$.



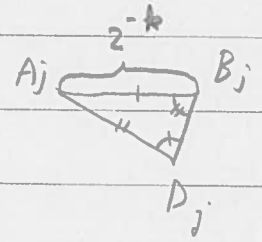
- Also,

$$h_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k+1} > \log(k+2)$$

$$\Rightarrow |R_j' \cap E| \geq \text{Area}(A_jB_jC_j)$$

(1) $$= \frac{1}{2} 2^{-k} \cdot h_k = \frac{1}{2^{k+1}} \log(k+2)$$

By law of sines,



$$|A_jD_j| = 2^{-k} \frac{\sin(\angle A_jB_jD_j)}{\sin(\angle A_jD_jB_j)}$$

(2) $$\leq \frac{2^{-k}}{\cos(\angle A_jC_jB_j)}$$

$$\left(\text{B/C } \angle C_jA_jD_j = 90^\circ,\ \sin(\angle A_jD_jB_j) = \cos(\angle A_jC_jD_j) = \cos(\angle A_jC_jB_j) \right)$$

· By law of cosines with $h_k \le |A_j C_j|, |B_j C_j| \le \frac{\sqrt{5}}{2} h_k$,

③ $$\cos(\angle A C_j B_j) \ge \frac{h_k^2 + h_k^2 - (2^{-k})^2}{2 \cdot \frac{5}{4} h_k^2} \ge \frac{4}{5} - \frac{2}{5} \cdot \frac{1}{4} \ge 1/2$$

b/c $\frac{1}{2^k h_k} < \frac{1}{2^k \log(k+2)} < 1/2$

· From ② & ③,

$$|A_j D_j| \le 2^{-k+1} = 2|A_j B_j|$$

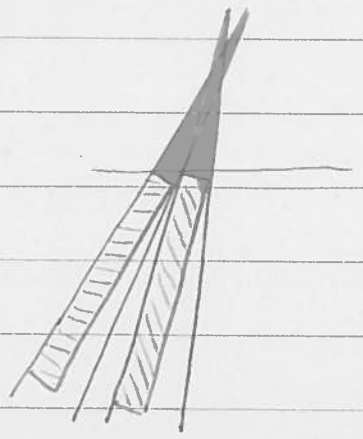
Then, from ①,

$$|R_i' \cap E| \ge \frac{1}{2^{k+1}} \log(k+2)$$

$$= \frac{1}{12} \underbrace{2^{-k+1}}_{\substack{\vee| \\ |A_j D_j|}} \cdot \underbrace{3 \log(k+2)}_{\text{long side}} \ge \frac{1}{12} |R_j|$$

$\Rightarrow$ (4).

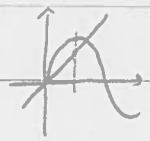
· (1) follows from the fact that the regions under $\angle A_j C_j B_j$ are disjoint since $\Delta A_j B_j C_j$ are pairwise disjoint.



- <u>Pf of (3)</u> From ②,

$$|A_jD_j| \geq 2^{-k}\sin(\angle A_jB_jD_j)$$

$$\geq 2^{-k-1}\angle A_jB_jD_j \qquad \left(\because \sin x \geq \frac{x}{2} \text{ on } [0, \tfrac{\pi}{2}]\right.$$



$$\geq 2^{-k-1}\angle B_jA_jC_j$$

$$\left(\because \begin{aligned} \frac{\pi}{2} &= \boxed{\angle B_jA_jD_j} + \angle B_jA_jC_j \\ \pi &= \boxed{\angle B_jA_jD_j} + \angle A_jB_jD_j + \underbrace{\angle A_jD_jB_j}_{<\pi/2} \end{aligned}\right.$$

and $\min_j \angle B_jA_jC_j = \angle B_0A_0C_0 = \tan^{-1}2 > 1$

$$\Rightarrow \quad |A_jD_j| \geq 2^{-k-1}$$

$$\Rightarrow \quad \text{Area}(R_j) \geq 2^{-k-1}\,3\log(k+2)$$

Hence,

$$\left|\bigcup_{j=0}^{2^k-1} R_j\right| = \sum_{j=0}^{2^k-1}|R_j| = 2^k\,2^{-k-1}\,3\log(k+2)$$

$$\geq |E|\log(k+2) \geq |E|/\delta$$

b/c $\begin{cases} |E| \leq 3/2 \\ k+2 \geq e^{1/\delta} \end{cases}$

$\blacksquare$

<u>Prop 2</u>: · R, rectangle whose center is at the origin in $\mathbb{R}^2$.

· v = unit vector parallel to the long side of R.

Consider the half plane

$$H = \{ x \in \mathbb{R}^2 : x \cdot v \geq 0 \}$$

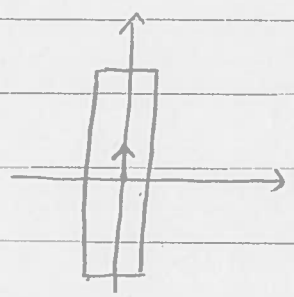
and the multiplier $S_H(f) = (\chi_H \hat{f})^{\vee}$.

Then, $|S_H(\chi_R)| \geq \frac{1}{10} \chi_{R'}$

<u>Remark</u>: The same conclusion holds for any rectangle R in $\mathbb{R}^2$ by applying translation.

(H = half-plane formed by the line passing through its center and parallel to its short side

<u>Pf</u>) WLOG, assume $R = [-a, a] \times [-b, b]$, $0 < a \leq b < \infty$ and $v = e_2 = (0, 1)$



Since F.T. acts on each variable independently, we have

$$S_H(\chi_R)(x,y) = \chi_{[-a,a]}(x)\left(\widehat{\chi_{[-b,b]}}\,\chi_{[0,\infty)}\right)^{\vee}(y)$$

$$= \chi_{[-a,a]}(x)\,\frac{I + iH}{2}\left(\chi_{[-b,b]}\right)(y)$$

where $H(f)(y) = \left((-i\,\mathrm{sgn}\,\xi)\hat{f}(\xi)\right)^{\vee}(x)$.

Hilbert transform.

$$\Rightarrow |S_H(\chi_R)(x,y)| \geq \frac{1}{2}\chi_{[-a,a]}(x)\left|H(\chi_{[-b,b]})(y)\right|$$

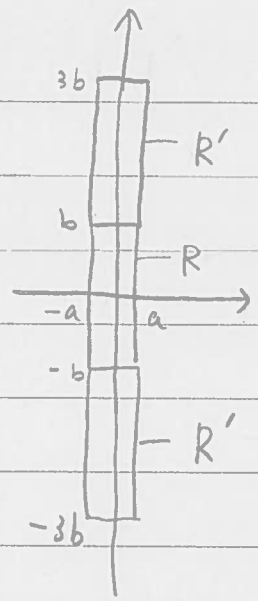
$$= \frac{1}{2\pi}\chi_{[-a,a]}(x)\left|\log\left|\frac{y+b}{y-b}\right|\right|$$

$$H(\chi_{[\alpha,\beta]})(x) = \frac{1}{\pi}\,\text{p.v.}\int \frac{\chi_{[\alpha,\beta]}(x-y)}{y}\,dy$$

$$= \frac{1}{\pi}\,\text{p.v.}\int_{\alpha}^{\beta}\frac{1}{y}\,dy$$

$$= \frac{1}{\pi}\lim_{\varepsilon\to 0}\left(\log\frac{|x-\alpha|}{\varepsilon} + \log\frac{\varepsilon}{|x-\beta|}\right)$$

$$= \frac{1}{\pi}\log\frac{|x-\alpha|}{|x-\beta|}$$



- For $(x, y) \in R'$,

  $\chi_{[-a,a]}(x) = 1$

  and $b < |y| < 3b$

- When $b < y < 3b$,

$$\left| \frac{y+b}{y-b} \right| = \frac{y+b}{y-b} > 2$$

- When $-3b < y < -b$, $\left| \frac{y-b}{y+b} \right| = \frac{b-y}{-b-y} > 2$

$\Rightarrow$ For $(x, y) \in R'$, we have

$$\left| S_H(\chi_R)(x,y) \right| \geq \frac{\log 2}{2\pi} \geq \frac{1}{10} \qquad \boxtimes$$

<u>Prop 3</u>: $\{v_j\}_{j \in \mathbb{N}}$, seq of unit vectors in $\mathbb{R}^2$.

Define the half-planes $H_j = \{ x \in \mathbb{R}^2 : x \cdot v_j \geq 0 \}$

and lin. operators $S_{H_j}(f) = (\chi_{H_j} \hat{f})^\vee$

- Assume the disc operator $T(f) = (\chi_B \hat{f})^\vee$ is bounded in L^1

Then,
$$\left\| \left(\sum_j |S_{H_j}(f_j)|^2 \right)^{1/2} \right\|_{L^p} \leq B_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p}$$

for all bounded, compactly supported f_j.

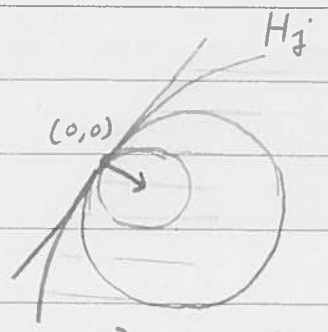
(Y. Meyer)

Pf) We'll prove this for $\{f_j\} \subset \mathcal{S}$.

Let $D_{j,R}$ be the discs $\{x \in \mathbb{R}^2 : |x - R v_j| \le R\}$

and let $T_{j,R}(f) = \left(\chi_{D_{j,R}} \hat{f}\right)^\vee$.

· Note that $\chi_{D_{j,R}} \to \chi_{H_j}$ ptwise as $R \to \infty$.



$\Rightarrow$ For $f \in \mathcal{S}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, we have

$$\lim_{R \to \infty} T_{j,R}(f)(x) = S_{H_j}(f)(x). \quad \text{by DCT.}$$

· By Fatou's lemma

$$\left\| \left(\sum_j |S_{H_j}(f_j)|^2 \right)^{1/2} \right\|_{L^p} \le \liminf_{R \to \infty} \left\| \left(\sum_j |T_{j,R}(f_j)|^2 \right)^{1/2} \right\|_{L^p}.$$

· Also, note that

$$T_{j,R}(f)(x) = e^{2\pi i R v_j \cdot x} T_R\left(e^{-2\pi i R v_j \cdot (\cdot)} f\right)(x)$$

where

$$T_R(f) = \left(\chi_{B(0,R)} \hat{f}\right)^\vee.$$

- Let $g_j = e^{-2\pi i R v_j \cdot (\cdot)} f_j$.

$$\Rightarrow \left\| \left(\sum_j |S_{H_j}(f_j)|^2 \right)^{1/2} \right\|_{L^p} \leq \liminf_{R \to \infty} \left\| \left(\sum_j |T_R(g_j)|^2 \right)^{1/2} \right\|_{L^p}$$

Since we assume $\|T(f)\|_{L^p} \leq B_p \|f\|_{L^p}$,

we have $\|T_R(f)\|_{L^p} \leq B_p \|f\|_{L^p}$.

$\because$ T has the multiplier $\chi_{B(0,1)}$ and T_R has the multiplier $\chi_{B(0,R)} = \delta^{1/R} \chi_{B(0,1)}$.

but $\|\delta^a(m)\|_{M_p} = \|m\|_{M_p}$

i.e. the norm of an L^p Fourier multiplier is invariant under dilation (translation, modulation, reflection, rotation.)

$\Rightarrow$ By ℓ^2-valued extension Theorem $(0 < p < \infty)$

$$\liminf_{R \to \infty} \left\| \left(\sum_j |T_R(g_j)|^2 \right)^{1/2} \right\|_{L^p}$$

$$\leq \liminf_{R \to \infty} \|T_R\|_{L^p \to L^p} \left\| \left(\sum_j |g_j|^2 \right)^{1/2} \right\|_{L^p}$$

$$= B_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p}$$

$\boxtimes$

<u>Theorem 1</u> χ_B in $\mathbb{R}^n$ is NOT an L^p multiplier when $1 < p \neq 2 < \infty$.

pf) Suffices to prove in $\mathbb{R}^2$ and $p > 2$

- Suppose $\chi_{B(0,1)} \in \mathcal{M}_p(\mathbb{R}^2)$ for some $p > 2$ with norm B_p.

- Let $\delta > 0$, and E, R_j be as in Lemma 1.

Let $f_j = \chi_{R_j}$, and let v_j be the unit vector parallel to the long side of R_j and H_j be the half-plane defined by v_j as before.

By Prop 2, $\displaystyle\int_E \sum_j |S_{H_j}(f_j)(x)|^2 dx$

$$\geq \sum_j \int_E \frac{1}{100} \chi_{R_j'}(x)\, dx$$

$$= \frac{1}{100} \sum_j |E \cap R_j'| \geq \frac{1}{1200} \sum_j |R_j| \quad \text{by (4) of Lemma 1.}$$

- On the other hand, by Hölder ineq with $p/2$ and $(p/2)' = \dfrac{p}{p-2}$, we have

$$\int_E \sum_j |S_{H_j}(f_j)(x)|^2 \, dx$$

$$\leq |E|^{\frac{p-2}{p}} \cdot \left\| \left(\sum_j |S_{H_j}(f_j)|^2 \right)^{1/2} \right\|_{L^p}^2$$

$$\leq B_p \, |E|^{\frac{p-2}{p}} \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p}^2$$

$$\leq B_p \, |E| \left(\sum_j |R_j| \right)^{2/p} \qquad (R_j, \text{ disjoint})$$

$$\leq B_p \, \delta^{\frac{p-2}{p}} \sum_j |R_j|$$

by (3) of lemma 1 : $|E| \leq \delta \sum_j |R_j|$

$$\left(R_j, \text{ disjoint} \Rightarrow \int \left(\sum_j \chi_{R_j} \right)^{p/2} dx \right.$$

$$= \int \sum_j \chi_{R_j} \, dx = \sum_j |R_j|.$$

Hence, $\sum_j |R_j| \leq 1200 \, B_p \, \delta^{\frac{p-2}{p}} \sum_j |R_j|$

$\Rightarrow$ (X) when δ is very small. $\boxtimes$

- <u>Appendix</u>:
  Note on ℓ^2-valued extensions of linear operators

<u>Theorem A1</u>: $0 < p, q < \infty$

(X, μ), (Y, ν), measure spaces

$T : L^p(X) \to L^q(Y)$, lin, bdd with $\|T\| = A$.

Then, T has an ℓ^2-valued extension
i.e $\forall f_j \in L^p(X)$, we have

$$(*) \quad \left\| \left(\sum_j |T(f_j)|^2 \right)^{1/2} \right\|_{L^q} \leq C_{p,q} A \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p}$$

for some $C_{p,q} = C(p,q)$.
Moreover, $C_{p,q} = 1$ if $p \leq q$.

- To prove this theorem, we need the following identities.

<u>Lemma A2</u>: For $0 < r < \infty$, let

$$A_r = \left(\frac{\Gamma\left(\frac{r+1}{2}\right)}{\pi^{\frac{r+1}{2}}} \right)^{1/r} \text{ and } B_r = \left(\frac{\Gamma\left(\frac{r}{2}+1\right)}{\pi^{r/2}} \right)^{1/r}.$$

Then, for any $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, we have

$$\text{(A)} \quad \left(\int_{\mathbb{R}^n} |\lambda_1 x_1 + \cdots + \lambda_n x_n|^r e^{-\pi|x|^2} dx \right)^{\frac{1}{r}} = A_r (\lambda_1^2 + \cdots + \lambda_n^2)^{1/2}$$

and for any $w_1, \cdots, w_n \in \mathbb{C}$, we have

$$\text{(B)} \quad \left(\int_{\mathbb{C}^n} |w_1 z_1 + \cdots + w_n z_n|^r e^{-\pi|z|^2} dz \right)^{1/r} = B_r (|w_1|^2 + \cdots + |w_n|^2)^{1/2}$$

· <u>Pf of lemma A2:</u> WLOG, assume $\sum_1^n \lambda_j^2 = 1$

Let $e_1 = (1, 0, \cdots, 0) \in S^{n-1}$

and $A \in O(n)$ s.t. $A^{-1} e_1 = (\lambda_1, \cdots, \lambda_n)$.

Then, the 1st coord of Ax is given by

$(Ax)_1 = Ax \cdot e_1 = x \cdot A^T e_1 = x \cdot A^{-1} e_1 = \sum_1^n \lambda_j x_j$.

· Now, by change of var $y = Ax$ in (A)

and $|Ax| = |x|$,

$$(\text{LHS}) \text{ of (A)} = \left(\int_{\mathbb{R}^n} |y_1|^r e^{-\pi|y|^2} dy \right)^{1/r}$$

$$= \left(2 \int_0^\infty t^r e^{-\pi t^2} dt \right)^{1/r} = \left(\int_0^\infty s^{\frac{r-1}{2}} e^{-\pi s} ds \right)^{\frac{1}{r}}$$

$$= \left(\frac{\Gamma\left(\frac{r+1}{2}\right)}{\pi^{\frac{r+1}{2}}} \right)^{1/r} = A_r.$$

$$\left(\because \int_{\mathbb{R}^{n-1}} e^{-\pi(y_2^2+\cdots+y_n^2)} dy_2 \cdots dy_n = \left(\int_{\mathbb{R}} e^{-\pi t^2} dt \right)^{n-1} = 1 \right.$$

- As for (B), assume $\sum_1^n |w_j|^2 = 1$.

Let $\varepsilon_1 = (1, 0, \cdots, 0) \in \mathbb{C}^n$,

and $A \in U(n)$ s.t. $A^{-1}\varepsilon_1 = (\overline{w_1}, \cdots, \overline{w_n})$

Then,

$$(Az)_1 = Az \cdot \overline{\varepsilon_1} = z \cdot \overline{A^{-1}\varepsilon_1} = \sum_1^n w_j z_j .$$

and $|Az| = |z|$.

The rest follows as before from change of var $\zeta = Az$.

$\boxed{/}$

<u>Pf of Thm A1</u>: If T maps real-valued functions to real-valued functions, then we may use (A) in Lemma A2.
In general, T maps $\mathbb{C}$-valued functions to $\mathbb{C}$-valued functions, and we use (B).

<u>case 1</u>: $q \le p$

Fix $m \in \mathbb{N}$. By successively using (B), the addness of T, Hölder ineq, and (B) again,

we have

$$\left\| \left(\sum_{j=1}^{n} T(f_j)^2 \right)^{1/2} \right\|_{L^q(Y)}^q$$

$$\underset{B}{=} (B_q)^{-q} \int_Y \int_{\mathbb{C}^n} |T(z_1 f_1 + \cdots + z_n f_n)|^q \, d\nu \, e^{-\pi|z|^2} dz$$

$$\leq (B_q)^{-q} A^q \int_{\mathbb{C}^n} \left(\int_X \left| \sum_{j=1}^{n} z_j f_j \right|^p d\mu \right)^{q/p} e^{-\pi|z|^2} dz$$

$$\leq (B_q)^{-q} A^q \left(\int_{\mathbb{C}^n} \int_X \left| \sum z_j f_j \right|^p d\mu \, e^{-\pi|z|^2} dz \right)^{q/p}$$

Hölder with $\frac{p}{q}$ and $\left(\frac{p}{q}\right)'$. i.e. need $\frac{p}{q} \geq 1$.

$$\underset{B}{=} (B_p B_q^{-1})^q A^q \left\| \left(\sum_{j=1}^{n} |f_j|^2 \right)^{1/2} \right\|_{L^p(X)}^q$$

Now, let $n \to \infty$ and obtain (*) with $C_{p,q} = B_p B_q^{-1}$.

<u>Case 2</u>: $q > p$ By a similar argument, we have

$$\left\| \left(\sum_{j=1}^{n} T(f_j)^2 \right)^{1/2} \right\|_{L^q}^q \leq A^q (B_q)^{-q} \int_{\mathbb{C}^n} \left(\int_X \left| \sum_{j=1}^{n} z_j f_j \right|^p d\mu \right)^{q/p} e^{-\pi|z|^2} dz.$$

$$\leq A^q (B_q)^{-q} \left\{ \int_X \left\| \left| \sum z_j f_j \right|^p \right\|_{L^{q/p}(\mathbb{C}^n, e^{-\pi|z|^2} dz)} d\mu \right\}^{q/p}$$

Minkowski integral ineq with $e^{-\pi|z|^2} dz$

$$\underset{B}{=} A^q B_q^{-q} \left\{ \int_X (B_q)^p \left(\sum_{j=1}^{n} |f_j|^2 \right)^{p/2} d\mu \right\}^{q/p} = A^q \left\| \left(\sum_{j=1}^{n} |f_j|^2 \right)^{1/2} \right\|_{L^p(X)}^q$$

$\square$

· <u>Remark</u>: ① As a corollary, we have

<u>Cor (ℓ^r-valued extension)</u>: $1 \le p < \infty$.

$T : L^p(X) \to L^p(Y)$, lin, bdd with $\|T\| = A$.

Then, for r bet p and 2, we have

$$\left\| \left(\sum_j |T(f_j)|^r \right)^{1/r} \right\|_{L^p} \le A \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_{L^p}.$$

pf) Clear from interpolation.

② ℓ^r-valued extension may fail if r does not lie between p and 2.

③ <u>Banach-valued extension</u>:

<u>Prop A3</u>: $0 < p, q \le \infty$, $(X, \mu), (Y, \nu)$ meas spaces

$T : L^p(X) \to L^q(Y)$, <u>positive</u>, lin, bdd with $\|T\| = A$.

$\mathcal{B}$. Banach space

Then, T has a $\mathcal{B}$-valued extension

$$\vec{T} \quad \text{via} \quad \vec{T}\left(\sum_{j=1}^n f_j u_j \right) = \sum_{j=1}^n T(f_j) u_j, \quad u_j \in \mathcal{B}$$

mapping $L^p(X, \mathcal{B})$ into $L^q(Y, \mathcal{B})$ with the same norm.

Lastly, we present a simple extension of Thm A1.

· <u>Prop A4:</u> $\mathcal{H}$, Hilbert space

$0 < p < \infty$

$T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, lin, bdd

Then, T has an $\mathcal{H}$-valued extension

In particular, $\forall$ measurable $\{f_t\}_{t \in \mathbb{R}^d}$,

$\forall$ positive meas μ on $\mathbb{R}^d$, we have

$$\left\| \left(\int_{\mathbb{R}^d} |T(f_t)|^2 d\mu(t) \right)^{1/2} \right\|_{L^p} \leq \|T\| \left\| \left(\int_{\mathbb{R}^d} |f_t|^2 d\mu(t) \right)^{1/2} \right\|_{L^p}.$$