The multiplier problem for the ball

\[ T, \text{ defined by } \hat{Tf}(\beta) = \hat{X_B}(\beta) \hat{f}(\beta) \]

where \( B = B(0,1) \).

Q: Is \( T \) bounded on \( L^p(\mathbb{R}^n) \)?

\[ \text{i.e. } \hat{X_B} \in \mathcal{M}_p(\mathbb{R}^n) \]

\[ \uparrow \text{ the set of } L^p \text{ Fourier multipliers} \]

\[ \mathcal{M}_p(\mathbb{R}^n) = \left\{ m \mid \text{Tm: } L^p \rightarrow L^p \right\} \]

\[ \text{where } \text{Tm}(f) = (m \hat{f})^\wedge, \ f \in \mathcal{S} \]

\[ \mathcal{M}^{p,p}(\mathbb{R}^n) = \{ T: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n) \text{ commuting with translations} \} \]

\[ \uparrow \text{ given by a convolution with a tempered distribution} \]

\[ \rightarrow m \in \mathcal{M}_p(\mathbb{R}^n) \iff \text{Tm} \in \mathcal{M}^{p,p}(\mathbb{R}^n) \]
Some basic facts:

1. $M_{p,q} = \{0\}$ if $1 \leq q < p \leq \infty$

2. $T \in M_{p,q} \iff T \in M_{p',q'}$

In particular, $M_p(\mathbb{R}^n) = M_{p'}(\mathbb{R}^n)$

3. $T \in M_{1,\infty} = M_{\infty,\infty}$$

$\iff T$ is given by convolution with a finite (C-valued) Borel measure $\mu$ (and $\|T\| = \|\mu\|$ total variation)

4. $T \in M^{2,2}$

$\iff T$ is given by convolution with some $\mu \in \mathcal{S}'$

$s.t. \ \hat{\mu} \in L^\infty \ \ (and \ \|T\| = \|\hat{\mu}\|_{L^\infty})$

5. $1 \leq p \leq q \leq 2,$

$$M_1 \subseteq M_p \subseteq M_q \subseteq M_2 = L^\infty$$

6. $M(\mathbb{R}^n, \mathbb{R}^m) \in M_p(\mathbb{R}^{n+m}).$ (st. f. a.e. $\mathbb{R}^n,$ the function $y \mapsto m(\hat{\mathbf{s}}, y)$ is in $M_p(\mathbb{R}^m)$ with

$$\|m(\hat{\mathbf{s}}, \cdot)\|_{M_p(\mathbb{R}^m)} \leq \|m\|_{M_p(\mathbb{R}^{n+m})}$$
In particular, if $X_B \in M_p(\mathbb{R}^2)$, then, $X_B \in M_p(\mathbb{R}^n)$, for $n \geq 3$.

**Known Results**

1. $T_{\text{cube}}$ is bounded on $L^p$, $1 < p < \infty$.

2. Let $f(x) = \begin{cases} 1, & |x| < 1/10 \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow$ If $f$ decays too slowly at $\infty$ and $T$ can NOT be bounded on $L^p(\mathbb{R}^n)$ unless $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

**Disc Conjecture:** $T$ is bounded on $L^p(\mathbb{R}^n)$ with $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.
Bochner–Riesz spherical summation operators

\[ T_s f (\xi) = \left( \max \left( 1 - |\xi|^2, 0 \right) \right)^s \hat{f}(\xi) \]

\[ T_s \text{ is bounded on } L^p (\mathbb{R}^2) \text{ for } s > 0 \]

where

\[
\frac{4}{\frac{3}{2s} + 2s} < p < \frac{4}{1 - 2s}
\]

In particular, for \( s = 0^+ \), we have

\[
\frac{4}{3} \leq p \leq \frac{11}{11 - \frac{2n}{n+1}} = \frac{2n}{n-1} \leftarrow \text{ exact range for disc conjecture}
\]

**More on (2)**

Recall: for \( g(x) = g_0(|x|) \) radial, we have

\[
G(\xi) = \frac{2\pi}{|\xi|^{n-2}} \int_0^\infty g_0(r) J_{\frac{n}{2} - 1} (2\pi r |\xi|) r^\frac{n}{2} dr
\]

Also,

\[
\frac{d G(\xi)}{d |\xi|} = \int_{S^{n-1}} e^{-2\pi i \xi \cdot \theta} d\theta = \frac{2\pi}{|\xi|^{n-2}} J_{\frac{n}{2} - 1} (2\pi |\xi|)
\]
With \( f(x) = X_{B(r, \frac{1}{10})}(x) \), we have

\[
Tf(x) = \int_{|y| < \frac{1}{10}} \hat{X}_B(x-y) \, dy
\]

is the average of \( \hat{X}_B \) around \( x \).

\[
\hat{X}_B(x) = \frac{2\pi}{|x|^{n-2}} \int_0^1 \frac{J_{n-1} \left( \frac{2\pi r|x|}{|x|^{n-2}} \right)}{r^{\frac{n}{2}}} \, dr
\]

(use \( \frac{d}{dZ} (Z^k J_k(Z)) = Z^k J_{k-1}(Z) \) for \( k > \frac{1}{2} \))

\[
= \frac{2\pi}{|x|^{\frac{n-2}{2}}} \left( \frac{1}{(2\pi|x|)^{\frac{n}{2}+1}} Z^{\frac{n}{2}} J_{\frac{n}{2}}(Z) \right) \frac{2\pi|x|}{2}
\]

Recall for \( k > -\frac{1}{2} \)

\[
J_k(r) = \left\{ \begin{array}{l}
\frac{r^k}{2^k \Gamma(k+1)} + O(r^{k+1}) \quad \text{as} \quad r \to 0 \\
\sqrt{\frac{2}{\pi r}} \cos \left( r - \frac{\pi k}{2} - \frac{\pi}{4} \right) + O(r^{-\frac{3}{2}}) \quad \text{as} \quad r \to \infty
\end{array} \right.
\]

\[
\Rightarrow \lim_{Z \to 0} Z^{\frac{n}{2}} J_{\frac{n}{2}}(Z) = 0
\]
\[ X_b(x) = \frac{J_{n/2}(2\pi |x|)}{|x|^{n/2}} \]

Similarly, with \( M_d(\delta) = \left[ \max (1 - |\delta|^2, 0) \right]^{d/2} \):
\[
M_d(\delta) = \frac{\Gamma(d+1)}{\pi^d} \frac{J_{n/2+d}(2\pi |x|)}{|x|^{n/2+d}}
\]

Hence, for \(|x|\) large,
\[
|Tf(x)| = \text{average of } X_b \text{ around } x
\]

\[
\sim \frac{1}{|x|^{n/2}} \frac{1}{\sqrt{x}} \frac{1}{|x|^{n+1/2}}
\]

which is in \( L^p(\mathbb{R}^n) \) for \( p > \frac{2n}{n+1} \)

not in \( L^p(\mathbb{R}^n) \) for \( p \leq \frac{2n}{n+1} \)

By duality, \( \left( p = \frac{2n}{n+1} \Rightarrow p' = \frac{2n}{n-1} \right) \)

we have \( p < \frac{2n}{n-1} \) if \( T \) could be bounded.
- It turned out that the disc conjecture is false.

**Theorem 1:** \( T \) is bounded only in \( L^2 \) (\( n > 1 \)).

Note that it is enough to disprove \( L^p \)-boundedness of \( T \) on \( \mathbb{R}^2 \) for \( p > 2 \).

- We begin with a certain geometric construction that at first sight has no apparent relationship with the multiplier problem for the ball.

Given \( \triangle ABC \), construct 2 triangles \( \triangle AMF \), \( \triangle BME \), called the *sprouts* of \( \triangle ABC \).

Given a geometric construction that at first glance has no apparent relationship with the multiplier problem for the ball.

Given \( \triangle ABC \), construct 2 triangles \( \triangle AMF \), \( \triangle BME \), called the *sprouts* of \( \triangle ABC \).

**Diagram:**

- \( b = AB \).
- \( h_0 = \) height of \( \triangle ABC \).
- \( M = \) midpoint of \( AB \).
- \( h_0 \), \( h_1 > h_0 \).

Let \( \text{Spr}(\triangle ABC) = \triangle AMF \cup \triangle BME \) be the *sprouted* figure obtained from \( \triangle ABC \).

- Clearly, \( \triangle ABC \subset \text{Spr}(\triangle ABC) \).
- Also, let "arms of the sprouted figure" = \( \text{Spr}(\triangle ABC) \setminus \triangle ABC \)
In this case, \( \text{Spr}(ABC) \) has two arms of equal area, \( \text{EGC} \) and \( \text{FCH} \).

Moreover,

\[
\text{Area (each arm of } \text{Spr}(ABC)) = \frac{b}{2} \left( h_1 - h_0 \right)^2 \frac{2}{2h_1 - h_0}
\]

By similarities of \( \triangle \)'s,

\[
\frac{NC}{b/2} = \frac{h_1 - h_0}{h_1}
\]

and

\[
\frac{\text{height(NGC)}}{h_0} = \frac{NC}{NC + b/2}
\]

Then, \( \text{height(EGC)} \)

\[
= h_0 \frac{NC}{NC + b/2} + h_1 - h_0 = \frac{2h_1 (h_1 - h_0)}{2h_1 - h_0}
\]

and \( \text{base(EGC)} = NC = \frac{h_1 - h_0}{h_1} \cdot \frac{b}{2} \)

\[
\Rightarrow |\triangle \text{EGC}| = \frac{b}{2} \left( h_1 - h_0 \right)^2 \frac{2}{2h_1 - h_0}
\]
Let $\Lambda = ABC$ be an isosceles triangle in $\mathbb{R}^2$ with base $AB$ of length $b_0 = \varepsilon$ and height $MC = h_0 = \varepsilon$.

Define the seq. of heights

$$h_1 = \left(1 + \frac{1}{2}\right) \varepsilon,$$

$$h_2 = \left(1 + \frac{1}{2} + \frac{1}{3}\right) \varepsilon,$$

$$h_j = \left(1 + \frac{1}{2} + \cdots + \frac{1}{j+1}\right) \varepsilon.$$

Apply the sprouting procedure to $\Lambda$ to obtain 2 sprouts $\Lambda_1 = AMF, \Lambda_2 = EMB$, each with height $h_1$ base length $b_0/2$.

Now, apply the same procedure to $\Lambda_1$ and $\Lambda_2$ and obtain

$\Lambda_{11}, \Lambda_{12}$ from $\Lambda_1$,

$\Lambda_{21}, \Lambda_{22}$ from $\Lambda_2$.

4 sprouts with height $h_2$. 
- Continue this process obtaining, at the $j$-th step, $2^j$ sprouts $\lambda_{r_1 \ldots r_j}$, $r_1, \ldots, r_j \in \{1, 2\}$, each with 
  \begin{align*}
  \text{base length } b_j &= 2^{-j} b_0, \\
  \text{height } h_j.
  \end{align*}

- Stop at the $k$-th step and let

$$E(\varepsilon, k) = \bigcup_{r_j \in \{1, 2\}} \lambda_{r_1 \ldots r_k}$$

$$\Rightarrow \text{Area } (E(\varepsilon, k)) \leq |\Lambda| + \text{Areas of arms of all the sprouted figures}$$

- At $j$-th step, each of $2^j$ arms has area

$$\frac{b_{j-1} \left( h_j - h_{j-1} \right)^2}{2 \left( 2h_j - h_{j-1} \right)}$$
\[ |E(\varepsilon, k)| \leq \frac{1}{2} \varepsilon^2 + \sum_{j=1}^{k} \frac{2^j - (j-1) b_0 \varepsilon^2}{2 (j+1)^2 \varepsilon} \]

\[ \leq \frac{1}{2} \varepsilon^2 + \sum_{j=1}^{\infty} \frac{\varepsilon^2}{j^2} \leq \left( \frac{1}{2} + \frac{1}{6} \right) \varepsilon^2 \leq \frac{3}{2} \varepsilon^2 \]

\[ \Rightarrow \text{For any } k, \ |E(\varepsilon, k)| \text{ can be made arbitrarily small if we take } \varepsilon \text{ small.} \]

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Formally, Given a rectangle \( R \subset R^2 \), let \( R' \) be two copies of \( R \) adjacent to \( R \) along its short side s.t. \( R \cup R' \) has the same width as \( R \) but 3 times its length.
Lemma 1: \( S > 0 \). Then, \( \mathcal{F} \) measurable \( E \subset \mathbb{R}^2 \) and a finite collection of rectangles \( R_j \) in \( \mathbb{R}^2 \)

s.t.
1. The \( R_j \)'s are pairwise disjoint
2. \( \frac{1}{2} \leq |E| \leq \frac{3}{2} \)
3. \( |E| \leq S \sum_j |R_j| \)
4. For all \( j \), \( |R_j \cap E| \geq \frac{1}{12} |R_j| \)

PF) Let \( ABC \) be an isosceles triangle of height 1 with \( A = (0,0) \) and \( B = (1,0) \).

- Given \( S > 0 \), choose \( k \in \mathbb{N} \)

s.t. \( k + 2 > e^{1/S} \)

For this \( k \), set \( E = E(1, k) \), the union of sprouted figures constructed earlier with \( E = 1 \).

Then, \( \frac{1}{2} \leq E \leq \frac{3}{2} \Rightarrow (2) \)

- Recall each \( [\frac{j}{2^k}, \frac{j+1}{2^k}] \) in \( [0, 1] \) is the base of exactly one sprouted triangle \( A_jB_jC_j \) (\( j = 0, 1, \ldots, 2^k - 1 \)).
Where \( A_j = \left( \frac{1}{2^k}, 0 \right) \), \( B_j = \left( \frac{1}{2^k} + \frac{1}{2^{k+1}}, 0 \right) \) and \( C_j \) is the other vertex of the sprouted triangle.

Define \( R_j \) inside \( \angle A_j C_j B_j \) as below.

\( R_j \) is defined s.t. one vertex is either \( A_j \) or \( B_j \) and the length of its long side is \( 3 \log (k+2) \).

Note that \( \max \frac{1}{2} \left( A_j C_j, B_j C_j \right) = \frac{1}{2} \cdot \frac{3 \log (k+2)}{2} \)
\[ \text{WLOG, assume } A_j C_j \geq B_j C_j. \]

Then,
\[ \frac{\sqrt{5}}{2} h_k < \frac{3}{2} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{k+1} \right) < \frac{3}{2} \left( 1 + \log(k+1) \right) < 3 \log(k+2). \]

\[ \therefore k \geq 1 \text{ and } e < 3 \Rightarrow 1 < \log(k+2) \]

Hence, \( R_j \) contains \( A_j B_j C_j \).

\[ \text{Also, } \]
\[ h_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k+1} > \log(k+2) \]

\[ \Rightarrow |R_j \cap E| = \text{Area}(A_j B_j C_j) \]

\[ = \frac{1}{2} 2^{-k} h_k = \frac{1}{2^{k+1}} \log(k+2) \]

By law of sines,
\[ |A_j D_j| = 2^{-k} \frac{\sin(\angle A_j B_j D_j)}{\sin(\angle A_j D_j B_j)} \]

\[ \leq \frac{2^{-k}}{\cos(\angle A_j C_j B_j)} \]

\[ (B/C \angle C_j A_j D_j = 90^{\circ}, \sin(\angle A_j D_j B_j) = \cos(\angle A_j C_j D_j) = \cos(\angle A_j C_j B_j)) \]
By law of cosines with \( h_k \leq |A_j C_j|, |B_j C_j| \leq \frac{\sqrt{5}}{2} h_k \),
\[
\cos(\angle A_j C_j B_j) = \frac{h_k^2 + h_k^2 - (2^{-k})^2}{2 \cdot \frac{5}{4} h_k^2} \geq \frac{4}{5} - \frac{2}{5} \cdot \frac{1}{4} \geq \frac{1}{2}
\]

From (2) & (3),
\[
|A_j D_j| \leq 2^{-k+1} = 2 \cdot |A_j B_j|.
\]

Then, from (1),
\[
|R'_1 \cap E| \geq \frac{1}{2^{k+1}} \log (k + 2)
\]
\[
= \frac{1}{12} \cdot 2^{-k+1} \cdot 3 \cdot \log (k + 2) \geq \frac{1}{12} \cdot \frac{1}{|A_j D_j|}
\]

\( \Rightarrow \) (4).

(1) follows from the fact that the regions under \( \angle A_j C_j B_j \) are disjoint since \( \triangle A_j B_j C_j \) are pairwise disjoint.
\[ \| A_j D_j \| \geq 2^{-k} \min (\angle A_j B_j D_j) \]
\[ \geq 2^{-k-1} \angle A_j B_j D_j \quad (\because \min x \geq \frac{x}{2} \text{ on } [0, \frac{\pi}{2}]) \]
\[ \geq 2^{-k-1} \angle B_j A_j C_j \]
\[ \frac{\pi}{2} = \left( \angle B_j A_j D_j + \angle B_j A_j C_j \right) \]
\[ \pi = \left( \angle B_j A_j D_j + \angle A_j B_j D_j + \angle A_j D_j B_j \right) < \frac{\pi}{2} \]

and \( \min \angle B_j A_j C_j = \angle B_0 A_0 C_0 = \tan^{-1} 2 > 1 \)

\[ \Rightarrow \| A_j D_j \| \geq 2^{-k-1} \]

\[ \Rightarrow \text{Area } (R_j) \geq 2^{-k-1} 3 \log (k+2) \]

Hence,
\[ \left| \bigcup_{j=0}^{2^{k-1}} R_j \right| = \sum_{j=0}^{2^{k-1}} \| R_j \| = 2^k 2^{-k-1} 3 \log (k+2) \]
\[ \geq |E| \log (k+2) \geq \frac{|E|}{\delta} \]

\[ |E| \leq \frac{3}{\delta} \]
\[ \begin{cases} \frac{\log (k+2)}{\delta} \geq e^{1/\delta} \end{cases} \]
Prop 2: \( R \), rectangle whose center is at the origin in \( \mathbb{R}^2 \).

- \( v \) = unit vector parallel to the long side of \( R \).
- Consider the half plane

\[ H = \{ x \in \mathbb{R}^2 : x \cdot v \geq 0 \} \]

and the multiplier \( S_H(f) = (X_H f)^v \).

Then, \( |S_H(X_R)| \geq \frac{1}{10} X_R \).

Remark: The same conclusion holds for any rectangle \( R \) in \( \mathbb{R}^2 \) by applying translation.

\( H \) = half-plane formed by the line passing through its center and parallel to its short side.

\[ \text{WLOG, assume } R = [-a, a] \times [-b, b], \quad 0 < a \leq b < \infty \]

and \( v = e_2 = (0, 1) \).
Since F.T acts on each variable independently,

\[ S_H(x_R)(x, y) = X^{(x)}_{[-a, a]} \left( X^{(y)}_{[-b, b]} X_{[0, \infty)} \right)^V(y) \]

\[ = X_{[-a, a]}(x) \sqrt{\frac{1 + iH}{2}} (X_{[-b, b]}(y) \]

where \[ H(f)(y) = \left( -\frac{1}{2\pi} \text{sgn} \frac{y}{\pi} \right) \hat{f}(\frac{y}{\pi}) \]

\[ \uparrow \text{Hilbert transform} \]

\[ \Rightarrow |S_H(x_R)(x, y)| \geq \frac{1}{2} X_{[-a, a]}(x) \left| H(X_{[-b, b]})(y) \right| \]

\[ = \frac{1}{2\pi} X_{[-a, a]}(x) \left| \log \left| \frac{y + b}{y - b} \right| \right| \]

\[ H(X_{[a, \beta]})(x) = \frac{1}{\pi} \text{p.v.} \int \frac{X_{[a, \beta]}(x - y)}{y} dy \]

\[ = \frac{1}{\pi} \text{p.v.} \int^{\beta}_a \frac{1}{y} dy \]

\[ = \frac{1}{\pi} \lim_{\varepsilon \to 0} \left( \log \frac{|x - \alpha|}{\varepsilon} + \log \frac{\varepsilon}{|x - \beta|} \right) \]

\[ = \frac{1}{\pi} \log \frac{|x - \alpha|}{|x - \beta|} \]
For \((x, y) \in \mathbb{R}', X(-a, a) (x) = 1\)

and \(b < |y| < 3b\)

- When \(-b < y < 3b\),

\[
\frac{y+b}{y-b} = \frac{y+b}{y-b} > 2
\]

- When \(-3b < y < -b\),

\[
\frac{y-b}{y+b} = \frac{b-y}{-b-y} > 2
\]

\[
\Rightarrow \text{ For } (x, y) \in \mathbb{R}', \text{ we have }
\]

\[
\left| S_H \left( X_{R'} \right)(x, y) \right| \geq \frac{\log^2}{2\pi} \geq \frac{1}{10} \quad \square
\]

Prop 3: \(\{v_j\}_{j \in \mathbb{N}}, \) seq of unit vectors in \(\mathbb{R}^2\).

Define the half-planes \(H_j = \{ x \in \mathbb{R}^2 : x \cdot v_j \geq 0 \}\)

and lin. operators \( S_{H_j} (f) = \left( X_{H_j} \hat{f} \right) \)

Assume the dual operator \(T(f) = \left( X_B \hat{f} \right) \) is bounded in \(L^p\)

Then,

\[
\left\| \left( \sum_{j} \left| S_{H_j} (f_j) \right|^2 \right)^{1/2} \right\|_{L^p} \leq B_p \left\| \left( \sum_{j} |f_j|^2 \right)^{1/2} \right\|_{L^p}
\]

for all bounded, compactly supported \(f_j\).

(Y. Meyer)
(pf) We'll prove this for $\{f_j\} < f$.

Let $D_{j, R}$ be the discs \( \{ x \in \mathbb{R}^2 : |x - Rv_j| \leq R \} \) and let $T_{j, R}(f) = (X_{D_{j, R}} \hat{f})^\vee$.

Note that $X_{D_{j, R}} \rightarrow X_{H_j}$ p.twise as $R \rightarrow \infty$.

\[ \Rightarrow \text{For } f \in L^1(\mathbb{R}^2) \text{ and } x \in \mathbb{R}^2, \text{ we have} \]

\[ \lim_{R \rightarrow \infty} T_{j, R}(f)(x) = \int_{H_j} (f)(x), \text{ by DCT.} \]

By Fatou's lemma

\[ \| (\sum_j |S_{H_j}(f_j)|^2)^{1/2} \|_{L^p} \leq \liminf_{R \rightarrow \infty} \| (\sum_j |T_{j, R}(f_j)|^2)^{1/2} \|_{L^p}. \]

Also, note that

\[ T_{j, R}(f)(x) = e^{2\pi i Rv_j \cdot x} T_R (e^{-2\pi i Rv_j \cdot \cdot (f)})(x) \]

where $T_R(f) = (X_{B(o, R)} \hat{f})^\vee$. 
Let \( g_j = e^{-2\pi i R v_j \cdot \cdot \cdot} f_j \).

\[ \Rightarrow \left\| \left( \sum_j |S_{H_j} (f_j) |^2 \right)^{1/2} \right\|_{L^p} \leq \lim \inf_{R \to \infty} \left\| \left( \sum_j |T_R (g_j) |^2 \right)^{1/2} \right\|_{L^p} \]

Since we assume \( \| T(f) \|_{L^p} \leq B_p \| f \|_{L^p} \),

we have \( \| T_R (f) \|_{L^p} \leq B_p \| f \|_{L^p} \).

\( \therefore \) \( T \) has the multiplier \( \chi_{B(0,1)} \) and \( T_R \) has the multiplier \( \chi_{B(0,R)} = \delta^R \chi_{B(0,1)} \).

but \( \| \delta^a (m) \|_{M_p} = \| m \|_{M_p} \)

i.e. the norm of an \( L^p \) Fourier multiplier is invariant under dilatation (translation, modulation, reflection, notation.)

\[ \Rightarrow \text{By } L^2 \text{-valued extension theorem } \left( 0 < p < \infty \right) \]

\[ \lim \inf_{R \to \infty} \left\| \left( \sum_j |T_R (g_j) |^2 \right)^{1/2} \right\|_{L^p} \]

\[ \leq \lim \inf_{R \to \infty} \| T_R \|_{L^p \to L^p} \| \left( \sum_j |g_j |^2 \right)^{1/2} \|_{L^p} \]

\[ = B_p \| \left( \sum_j |f_j |^2 \right)^{1/2} \|_{L^p} \]
Theorem 1. \( X_B \) in \( \mathbb{R}^n \) is NOT an \( L^p \) multiplier when \( 1 < p + 2 < \infty \).

pf) Suffices to prove in \( \mathbb{R}^2 \) and \( p > 2 \).

- Suppose \( X_{B(0,1)} \in \mathcal{M}_p(\mathbb{R}^2) \) for some \( p > 2 \) with norm \( B_p \).

  - Let \( J > 0 \), and \( E, R_j \) be as in Lemma 1.

  - \( f_j = X_{R_j} \), and let \( V_j \) be the unit vector parallel to the long side of \( R_j \) and \( H_j \) be the half-plane defined by \( V_j \) as before.

By Prop 2, \[
\int_E \sum_j \left| S_{H_j} (f_j)(x) \right|^2 \, dx
\]
\[
\geq \sum_j \int_E \frac{1}{100} X_{R_j'}(x) \, dx
\]
\[
= \frac{1}{100} \sum_j |E \cap R_j'| \geq \frac{1}{1200} \sum_j |R_j| \text{ by (4) of Lemma 1.}
\]

On the other hand, by Hölder ineq with \( p/2 \) and \( (p/2)' = \frac{p}{p-2} \), we have
\[ \int_E \sum_j |S_{H_j}(f_j)(x)|^2 \, dx \leq |E|^{p-2} \| (\sum_j |S_{H_j}(f_j)|^2)^{1/2} \|_{L^p}^2 \]
\[ \leq B_p |E|^{p-2} \| (\sum_j |f_j|^2)^{1/2} \|_{L^p}^2 \]
\[ \leq B_p |E| \left( \sum_j |R_j| \right)^{2/p} \text{ (R_j, disjoint)} \]
\[ \leq B_p \Delta^{p-2} \sum_j |R_j| \]

by (3) of lemma 1: \[ |E| \leq \delta \sum_j |R_j| \]

\[ \left( R_j, \text{ disjoint} \Rightarrow \int (\sum_j \chi_{R_j})^{p/2} \, dx \right) \]
\[ = \int \sum_j \chi_{R_j} \, dx = \sum_j |R_j| \]

Hence, \[ \sum_j |R_j| \leq 1200 B_p \Delta^{p-2} \sum_j |R_j| \]

\[ \Rightarrow \bigodot \text{ when } \delta \text{ is very small. } \]
Appendix:

Note on $\ell^2$-valued extensions of linear operators

**Theorem A1:** $0 < p, q < \infty$

$(X, \mu), (Y, \nu)$, measure spaces

$T: L^p(X) \to L^q(Y)$, lin, odd with $\|T\| = A$.

Then, $T$ has an $\ell^2$-valued extension

i.e. $\forall f \in L^p(X)$, we have

\[
\bigg(\sum_j |T(f_j)|^2\bigg)^{1/2} \leq C_{p, q} A \bigg(\sum_j |f_j|^2\bigg)^{1/2}
\]

\[
\| (\sum_j |T(f_j)|^2)^{1/2} \|_{L^q} \leq C_{p, q} A \| (\sum_j |f_j|^2)^{1/2} \|_{L^p}
\]

for some $C_{p, q} = C(p, q)$.

Moreover, $C_{p, q} = 1$ if $p = q$.

To prove this theorem, we need the following identities.

**Lemma A2:** For $0 < r < \infty$, let

\[
A_r = \left(\frac{\Gamma\left(\frac{r+1}{2}\right)}{\pi^{\frac{r+1}{2}}}\right)^{1/r}
\]

and

\[
B_r = \left(\frac{\Gamma\left(\frac{r+1}{2}\right)}{\pi^{\frac{r+1}{2}}}\right)^{1/r}
\]
Then, for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, we have
\[ \left( \int_{\mathbb{R}^n} |\lambda_1 x_1 + \cdots + \lambda_n x_n|^r e^{-\pi |x|^2} dx \right)^{1/r} = A_r \left( \lambda_1^2 + \cdots + \lambda_n^2 \right)^{1/2} \]

and for any $w_1, \ldots, w_n \in \mathbb{C}$, we have
\[ \left( \int_{\mathbb{C}^n} |w_1 z_1 + \cdots + w_n z_n|^r e^{-\pi |z|^2} dz \right)^{1/r} = B_r \left( |w_1|^2 + \cdots + |w_n|^2 \right)^{1/2} \]

\[ \text{Pf of lemma A2: WLOG, assume } \sum_{i=1}^n \lambda_i^2 = 1 \]

Let $e_i = (1, 0, \ldots, 0) \in S^{n-1}$

and $A \in O(n)$ s.t. $A^t e_i = (\lambda_i, \ldots, \lambda_n)$.

Then, the $1^\text{st}$ coord of $Ax$ is given by
\[ (Ax)_1 = A x \cdot e_i = x \cdot A^t e_i = x \cdot A^t e_i = \sum_{i=1}^n \lambda_i x_i. \]

Now, by change of var $y = Ax$ in (A) and $|Ax| = |x|$, we have
\[ \text{(LHS) of (A)} = \left( \int_{\mathbb{R}^n} |y_1|^r e^{-\pi |y|^2} dy \right)^{1/r} \]
\[ = \left( 2 \int_0^\infty t^{r-1} e^{-\pi t^2} dt \right)^{1/r} = \left( \int_0^\infty s^{r-1} e^{-\pi s} ds \right)^{1/r} \]
\[ = \left( \frac{\Gamma \left( \frac{r+1}{2} \right)}{\pi^{r/2}} \right)^{1/r} = A_r. \]
\[
\left( - \int_{\mathbb{R}^{n-1}} e^{-\pi (y_2^2 + \cdots + y_n^2)} \, dy_2 \cdots dy_n = \left( \int_{\mathbb{R}} e^{-\pi t^2} \, dt \right)^{n-1} \right)
\]

As for (B), assume \( \sum_{j=1}^{n} |w_j|^2 = 1 \).

Let \( E_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n \), and \( A \in U(n) \) s.t. \( A^{-1} E_1 = (\overline{w}_1, \ldots, \overline{w}_n) \).

Then,
\[
(Az) \cdot E_1 = Az \cdot \overline{E}_1 = z \cdot A^{-1} E_1 = \sum_{j=1}^{n} w_j z_j,
\]
and \( |Az| = |z| \).

The rest follows as before from change of var \( \Sigma = Az \).

\( \square \)

**Pf of Thm A1:** If \( T \) maps real-valued functions to real-valued functions, then we may use (A) in Lemma A2.

In general, \( T \) maps \( \mathbb{C} \)-valued functions to \( \mathbb{C} \)-valued functions, and we use (B).

**Case 1:** \( q \leq p \)

Fix \( m \in \mathbb{N} \). By successively using (B), the boundedness of \( T \), Hölder inequality, and (B) again, we have
\[ \left\| \left( \sum_{j=1}^{n} T(f_j)^2 \right)^{\frac{q}{2}} \right\|_{L^q(Y)} \]

\[ = (B_q)^{-q} \int \int \frac{1}{C^n} T(Z_1 f_1 + \cdots + Z_n f_n)^q d\mu e^{-\pi |z|^2} dz \]

\[ \leq (B_q)^{-q} A^q \left( \int_X \left( \sum_{j=1}^{n} \left| Z_j f_j \right|^2 \right)^{\frac{q}{p}} d\mu \right)^{\frac{q}{p}} e^{-\pi |z|^2} dz \]

\[ \leq (B_q)^{-q} A^q \left( \int_X \left( \sum_{j=1}^{n} \left| Z_j f_j \right|^2 \right)^{\frac{q}{p}} d\mu \right)^{\frac{q}{p}} \]

\[ \text{Hölder with } \frac{p}{q} \text{ and } \left( \frac{p}{q} \right)' \text{, i.e. need } \frac{p}{q} \geq 1. \]

\[ = (B_p B_q^{-1})^{-q} A^q \left( \sum_{j=1}^{n} |f_j|^2 \right)^{\frac{q}{p}} \left\| L^p(X) \right\| \]

Now, let \( n \to \infty \) and obtain \( \circ \) with \( C_p, q = B_p B_q^{-1} \).

\[ \text{Case 2: } q > p \quad \text{By a similar argument, we have} \]

\[ \left\| \left( \sum_{j=1}^{n} T(f_j)^2 \right)^{\frac{q}{2}} \right\|_{L^q(Y)} \]

\[ \leq A^q (B_q)^{-q} \int \int \left( \sum_{j=1}^{n} \left| Z_j f_j \right|^2 \right)^{\frac{q}{p}} d\mu e^{-\pi |z|^2} dz \]

\[ \leq (B_q)^{-q} A^q \left\{ \int_X \left( \sum_{j=1}^{n} |f_j|^2 \right)^{\frac{q}{p}} d\mu \right\}^{\frac{q}{p}} \]

\[ \text{Minkowski integral meq with } e^{-\pi |z|^2} dz \]

\[ = A^q B_q^{-q} \left\{ \int_X (B_q)^p \left( \sum_{j=1}^{n} |f_j|^2 \right)^{\frac{q}{p}} d\mu \right\}^{\frac{q}{p}} \]

\[ \text{Case 2: } q > p \quad \text{By a similar argument, we have} \]

\[ \left\| \left( \sum_{j=1}^{n} T(f_j)^2 \right)^{\frac{q}{2}} \right\|_{L^q(Y)} \]

\[ \leq (B_q)^{-q} A^q \left( \sum_{j=1}^{n} |f_j|^2 \right)^{\frac{q}{p}} \left\| L^p(X) \right\| \]

\[ \square \]
Remark: (1) As a corollary, we have

\text{Cor \ (L^r-valued extension): \ 1 \leq p < \infty.}

\[ T: L^p(X) \to L^p(Y), \text{ lin, add with } \|T\| = A. \]

Then, for \( r \) bet \( p \) and \( 2 \), we have

\[ \| (\sum_j |T(f_j)|^r)^{\frac{1}{r}} \|_{L^p} \leq A \| (\sum_j |f_j|^r)^{\frac{1}{r}} \|_{L^p}. \]

\textbf{pf:} Clear from interpolation.

(2) \( L^r \)-valued extension may fail if \( r \) does not lie between \( p \) and \( 2 \).

(3) \underline{Banach-valued extension:}

Proposition: \( 0 < p, q \leq \infty \). \((X, \mu), (Y, \nu)\) measurable spaces

\[ T: L^p(X) \to L^q(Y), \text{ positive, lin, add with } \|T\| = A. \]

B. Banach space

Then, \( T \) has a \( B \)-valued extension

\[ \hat{T} \text{ via } \hat{T}(\sum_{j=1}^n f_j u_j) = \sum_{j=1}^n T(f_j) u_j, \quad u_j \in B \]

mapping \( L^p(X, B) \) into \( L^q(Y, B) \) with the same norm.
Lastly, we present a simple extension of Thm A1.

Prop A4: \( H \), Hilbert space

\[ 0 < p < \infty \]

\[ T : L^p (\mathbb{R}^d) \to L^p (\mathbb{R}^d) \text{, lin, bdd} \]

Then, \( T \) has an \( H \)-valued extension.

In particular, \( \mathcal{F} \) measurable \( \{ f_t \}_{t \in \mathbb{R}^d} \),

\( \mathcal{F} \) positive meas. \( \mu \) on \( \mathbb{R}^d \), we have

\[ \left\| \left( \int_{\mathbb{R}^d} |T(f_t)|^2 \, d\mu(t) \right)^{1/2} \right\|_{L^p} \leq \| T \| \left( \int_{\mathbb{R}^d} |f_t|^2 \, d\mu(t) \right)^{1/2} \right\|_{L^p}. \]