

Stochastic dispersive dynamics

Chapter 3: On the transport property of Gaussian measures under Hamiltonian PDE dynamics

Tadahiro Oh

The University of Edinburgh

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Hausdorff School: Dispersive Equations, Solitons, and Blow-up

Gaussian measures μ_s :

$$“d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^d} e^{-\frac{1}{2}\langle n \rangle^{2s} |\hat{u}_n|^2} d\hat{u}_n”$$

- μ_s is a Gaussian probability measure on $H^\sigma(\mathbb{T}^d)$ for $\sigma < s - \frac{d}{2}$
- Under μ_s , a random function u is represented by the random Fourier series:

$$u(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^s} e^{2\pi i n \cdot x} \in H^\sigma(\mathbb{T}^d) \setminus H^{s-\frac{d}{2}}(\mathbb{T}^d), \text{ almost surely}$$

where $\{g_n(\omega)\}_{n \in \mathbb{Z}^d} =$ independent standard \mathbb{C} -valued Gaussian r.v.'s

Let $\vec{\mu}_s = \mu_s \otimes \mu_{s-1}$

$\implies \vec{\mu}_s$ is **invariant** under the linear wave dynamics for any $s \in \mathbb{R}$

Goal:

Study transport properties of **Gaussian measures** μ_s (or $\vec{\mu}_s$) under *nonlinear* Hamiltonian PDE dynamics

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(H, B, μ) , abstract Wiener space, $H =$ Cameron-Martin space

Cameron-Martin Theorem: Consider the following translation map:

$$T_h : u \mapsto u + h \quad \text{for some } h \in B$$

Q: What is the relation between the original Gaussian measure μ on B and the translated measure $\mu^h(\cdot) = (T_h)_*\mu(\cdot) = \mu(\cdot - h)$?

Theorem: Cameron-Martin '44

- If $h \in H$, μ and μ^h are equivalent (= mutually absolutely continuous).
Namely, μ is **quasi-invariant** under T_h
- Otherwise, they are mutually singular

- This allows us to take a derivative of μ in the direction of $h \in H$ (= H -derivative)
 \implies starting point of Malliavin calculus
- For μ_s on $\mathcal{D}'(\mathbb{T}^d)$, μ_s and μ_s^h are equivalent if and only if $h \in H^s(\mathbb{T}^d)$.
Namely, h is $(\frac{d}{2} + \varepsilon)$ -**smoother** than typical elements $u \in H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$

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Ramer's generalization of Cameron-Martin Theorem:

$$T : u \mapsto u + F(u)$$

- We now allow the shift $F(u)$ to depend on a random element $u \in B$

Theorem: Ramer '74 (rough formulation)

μ is quasi-invariant under T if the H -derivative of F at u :

$$DF(u) : H \rightarrow H$$

is a Hilbert-Schmidt map for every $u \in B$

- For μ_s on $\mathcal{D}'(\mathbb{T}^d)$, (roughly speaking)
 μ_s is quasi-invariant under T if $F(u)$ is $(d + \varepsilon)$ -smoother than $u \in H^\sigma(\mathbb{T}^d)$
(More smoothing than Cameron-Martin Theorem, now that the shift is random)

There are also works on quasi-invariance of μ under flows generated by vector fields:
Cruzeiro '83, Peters '95, Bogachev and Mayer-Wolf '99, Ambrosio-Figalli '09, etc.

Duhamel formulation (for NLW): With $\vec{S}(t) = \begin{pmatrix} \cos(tD) & \frac{\sin(tD)}{D} \\ -D \sin(tD) & \cos(tD) \end{pmatrix}$,

$$\begin{aligned} \vec{u} &= \Phi(t)\vec{u}(0) = \vec{S}(t)\vec{u}(0) - \int_0^t \vec{S}(t-t') \begin{pmatrix} 0 \\ u^k(t') \end{pmatrix} dt' \\ &= \vec{S}(t) \left\{ \vec{u}(0) + \underbrace{\int_0^t \vec{S}(-t') \begin{pmatrix} 0 \\ u^k(t') \end{pmatrix} dt'}_{=F(\vec{u}(0))} \right\} \end{aligned}$$

- Gaussian measure μ_s is *invariant* under the linear solution map $\vec{S}(t)$
 \implies The solution map $\Phi(t)$ is of the form “ $\vec{u}(0) + F(\vec{u}(0))$ ”

Q: Can we study transport properties (such as invariance, quasi-invariance, singularity) of μ_s under nonlinear dispersive Hamiltonian PDEs?

Goal: Investigate transport properties of Gaussian measure μ_s :

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du \quad \text{on } H^\sigma(\mathbb{T}^d), \quad \sigma < s - \frac{d}{2}$$

- $s = 0$: **White noise** on \mathbb{T} : very rough

$$u(x) = \sum_{n \in \mathbb{Z}} g_n(\omega) e^{inx} \in H^\sigma(\mathbb{T}) \setminus H^{-\frac{1}{2}}(\mathbb{T}), \quad \sigma < -\frac{1}{2}$$

Invariance of white noise

- KdV: Quastel-Valkó '08, Oh '09, '11, Oh-Quastel-Valkó '12
- (renormalized) cubic fourth order NLS (4NLS): Oh-Tzvetkov-Wang '17

Q: Is white noise μ_0 invariant under (renormalized) cubic NLS on \mathbb{T} ?

- **Very difficult**
- μ_0 is a limit of invariant measures for cubic NLS (Oh-Quastel-Valkó '12) but no well-posedness...

Q: Can we study transport properties of μ_s for general (non-small) s ?

- When s is large, this question is *not* about rough solutions

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Quasi-invariance of Gaussian measures μ_s

- ($s = 0$) white noise μ_0 on \mathbb{T} : KdV and (Wick ordered) 4NLS invariance \implies quasi-invariance
- ($s = 1$) invariant Gibbs measure (= Gaussian measure μ_1 with weight) $\implies \mu_1$ is quasi-invariant
- completely integrable PDEs with infinitely many conservation laws \implies invariant measures ρ_k (= μ_k with weight) for every integer $k \geq 2$
 - cubic NLS on \mathbb{T} , KdV on \mathbb{T} , Benjamin-Ono equation on \mathbb{T} (Zhidkov '01, Tzvetkov-Visciglia '14-15, Deng-Tz-V '15)
 - derivative NLS on \mathbb{T} : open (only construction)

Q: Gel'fand '96: Can we directly prove quasi-invariance of μ_s (at least for s large) for (non-integrable) PDEs?

Remark:

- Gibbs measure problem: study of rough solutions
- When s is large, this question is *not* about rough solutions

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Benjamin-Bona-Mahony equation (BBM) on \mathbb{T} : small amplitude long surface waves

$$\begin{aligned}\partial_t u + \partial_x u - \partial_t \partial_x^2 u + \partial_x(u^2) &= 0 \\ \implies \partial_t u + (1 - \partial_x^2)^{-1} \partial_x u + (1 - \partial_x^2)^{-1} \partial_x(u^2) &= 0\end{aligned}$$

- **Ramer's result:** μ_s on $\mathcal{D}'(\mathbb{T})$ is quasi-invariant under the map

$$T : u_0 \mapsto u_0 + F(u_0)$$

if $F(u_0)$ is $(d + \varepsilon)$ -smoothing \implies not sufficient for BBM

Tzvetkov '15: For $s \in \mathbb{N}$, μ_s is quasi-invariant under BBM

- A similar result holds for generalized BBM model with less smoothing
- introduced a new method to establish quasi-invariance of μ_s beyond Ramer
- uses the **explicit smoothing** in the nonlinearity but *not* dispersive effect

Q: Can we find a good model to prove quasi-invariance via *dispersive effect*?

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Q: Can we find a good model to prove quasi-invariance via *dispersive effect*?

Cubic fourth order NLS (4NLS) on \mathbb{T} :

$$i\partial_t u - \partial_x^4 u = |u|^2 u$$

- Globally well-posed in $H^\sigma(\mathbb{T})$, $\sigma \geq 0$
- Strongly ill-posed for $\sigma < 0$ (Oh-Wang '17: non-existence in negative Sobolev spaces)

Theorem: Oh-Tzvetkov '16, Oh-Sosoe-Tzvetkov '17

Let $s > \frac{1}{2}$. Then, the Gaussian measure μ_s is quasi-invariant under 4NLS

- This theorem is *optimal*: μ_s is supported on $H^\sigma(\mathbb{T})$, $\sigma < s - \frac{1}{2}$
- Unlike BBM, there is *no* apparent smoothing in 4NLS. We exhibit smoothing effects *via dispersion* after using some *gauge transform* and *normal form reductions*
- The proof consists of *local & global analysis* (in the phase space $H^\sigma(\mathbb{T})$)
 - local PDE analysis (normal form reductions, energy estimates)
 - global phase space analysis (gauge transform, a change-of-variable formula)

Key role of dispersion

Q: Is *dispersion* essential for quasi-invariance of μ_s ?

Yes. Consider the dispersionless model on \mathbb{T} :

$$i\partial_t u = |u|^2 u$$

- Explicit solution formula $u(t, x) = e^{-it|u(0,x)|^2} u(0, x)$
- Globally well-posed in $H^\sigma(\mathbb{T})$, $\sigma > \frac{1}{2}$

Note: our random data u is a.s. continuous for $s > \frac{1}{2} \implies \sigma > 0$

Theorem: Oh-Sosoe-Tzvetkov '17

Let $s > \frac{1}{2}$. Then, μ_s is *not* quasi-invariant under the dispersionless model

- The proof uses law of iterated logarithms, a fine criterion to measure the regularity of a typical function w.r.t. μ_s (= fractional Brownian loop). This property regularity property is destroyed by the flow of the dispersionless model

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Rough idea

Goal : Compute $\frac{d}{dt}\mu_s(\Phi(t)(A))$

① **Energy estimate** (local analysis): $\frac{d}{dt}\|\Phi(t)(u)\|_{H^s}^2 \leq \underbrace{C(\|u\|_{L^2})}_{\text{conserved}} \underbrace{\|u\|_{H^{s-\frac{1}{2}-\varepsilon}}^{2-\theta}}_{\text{supp}(\mu_s)}$

② **A change-of-variable formula** (global analysis):

$$\mu_s(\Phi(t)(A)) = Z_s^{-1} \int_{\Phi(t)A} e^{-\frac{1}{2}\|u\|_{H^s}^2} du \quad \text{"="} \quad Z_s^{-1} \int_A e^{-\frac{1}{2}\|\Phi(t)(u)\|_{H^s}^2} du$$

\implies (Yudovich) Given $t \in \mathbb{R}$ and $\delta > 0$, there exists $C = C(t, \delta) > 0$ such that

$$\mu_s(\Phi(t)(A)) \leq C(t, \delta) \{\mu_s(A)\}^{1-\delta}$$

\implies quasi-invariance!!

- In Step 1, we need to apply two transformations on the phase space. Then, perform (an infinite iteration of) normal form reductions to prove the energy estimate on a *modified* energy $E = \|u\|_{H^s}^2 + R$
- In Step 2, we need to insert the frequency truncation $\mathbf{P}_{\leq N}$. Moreover, we need to consider a *modified* measure associated to the modified energy

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Decomposition of solution map:

$$\Phi(t) = \mathcal{G}_{-t} \circ S(t) \circ \Psi(t)$$

- ① **Gauge transform:** Given $t \in \mathbb{R}$, define \mathcal{G}_t on $L^2(\mathbb{T})$ by setting

$$\mathcal{G}_t[f] := e^{it \int_{\mathbb{T}} |f|^2} f$$

- ② **Interaction representation:** $v(t) = S(-t)\mathcal{G}_t[u(t)]$, where $S(t) = e^{-it\partial_x^4}$

- $\Phi(t)$ = solution map of the original 4NLS
- $\Psi(t)$ = solution map of $v(0) \mapsto v(t) = S(-t)\mathcal{G}_t[u(t)]$

Proposition

Let $s > \frac{1}{2}$. For every $t \in \mathbb{R}$, the Gaussian measure μ_s is invariant under $S(-t)$ and \mathcal{G}_t

\implies Suffices to prove quasi-invariance of μ_s under $\Psi(t)$

Local analysis: modified energy and energy estimate

- $v = S(-t) \circ \mathcal{G}_t[u(t)]$ satisfies

$$\partial_t \widehat{v}_n = -i \sum_{\{\phi(\bar{n}) \neq 0\}} e^{-i\phi(\bar{n})t} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} + i|\widehat{v}_n|^2 \widehat{v}_n$$

On $\Gamma(n) \stackrel{\text{def}}{=} \{\phi(\bar{n}) \neq 0\}$, we have $|\phi(\bar{n})| \gtrsim n_{\max}^2 \iff$ rapid oscillation

- **Modified energy:** $E(v) = \|v\|_{H^s}^2 + R(v) \leftarrow$ correction term

Proposition: energy estimate with smoothing

Let $s > \frac{3}{4}$. Then, for any small $\varepsilon > 0$, there exist $\theta > 0$ and $C > 0$ such that

$$\left| \frac{d}{dt} E(\mathbf{P}_{\leq N} v) \right| \leq \underbrace{C(\|v\|_{L^2})}_{\text{conserved}} \underbrace{\|v\|_{H^{s-\frac{1}{2}-\varepsilon}}^{2-\theta}}_{\text{supp}(\mu_s)}$$

- (*Infinite* iteration of) **normal form reductions** \rightarrow correction term R
- Standard (deterministic) PDE analysis
- The proof relies on elementary number theory (divisor counting argument)

By normal form reduction (IBP in time), we have

$$\begin{aligned}
 \frac{d}{dt} \|v(t)\|_{H^s}^2 &= -2 \operatorname{Re} i \sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} e^{-i\phi(\bar{n})t} \langle n \rangle^{2s} v_{n_1} \overline{v_{n_2}} v_{n_3} \overline{v_n} \\
 &= -2i \operatorname{Re} \underbrace{\frac{d}{dt} \left[\sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n \rangle^{2s} v_{n_1} \overline{v_{n_2}} v_{n_3} \overline{v_n} \right]}_{=: -R(v)} \\
 &\quad + 2i \operatorname{Re} \sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n \rangle^{2s} \underbrace{\partial_t (v_{n_1} \overline{v_{n_2}} v_{n_3} \overline{v_n})}_{=6\text{-linear}}
 \end{aligned}$$

When $s \in (\frac{1}{2}, \frac{3}{4}]$, iterate this process *infinitely many times*:

$$\begin{aligned}
 \frac{d}{dt} \|v(t)\|_{H^s}^2 &= \frac{d}{dt} \left[\underbrace{\sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(v)}_{=: -R(v)} \right] + \sum_{j=2}^{\infty} \mathcal{N}_1^{(j)}(v) + \sum_{j=2}^{\infty} \mathcal{R}^{(j)}(v) \\
 \implies \left| \frac{d}{dt} E(\mathbf{P}_{\leq N} v) \right| &\leq C(\|v\|_{L^2}), \quad \text{where } E(v) = \|v\|_{H^s}^2 + R(v)
 \end{aligned}$$

- **Guo-Kwon-Oh '13**: infinite iteration of NF reductions for cubic NLS on \mathbb{T} (i.e. on the equation) in the context of low regularity uniqueness problem

1 Weighted Gaussian measures: $E(v) = \|v\|_{H^s}^2 + R(v)$

- Construct a weighted Gaussian measure $\rho_{s,N,r}$ of the form:

$$\begin{aligned} d\rho_{s,N,r} &= Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}E(\mathbf{P}_{\leq N}v)} dv \\ &= Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}R(\mathbf{P}_{\leq N}v)} \underbrace{e^{-\frac{1}{2}\|v\|_{H^s}^2} dv}_{d\mu_s} \end{aligned}$$

2 A change-of-variable formula:

$$\rho_{s,N,r}(\Psi_N(t)(A)) = \hat{Z}_{s,N,r}^{-1} \int_A \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}E(\mathbf{P}_{\leq N}\Psi_N(t)(v))} d(\mathbf{P}_{\leq N}v) \otimes d\mu_{s,N}^\perp$$

3 Study **measure evolution** & take limits ($N \rightarrow \infty$, then $r \rightarrow \infty$)

- compute time derivative (**energy estimate**)

\implies quasi-invariance of $\rho_{s,N,r}$ under $\Psi_N(t)$

$\xrightarrow{N \rightarrow \infty} \implies$ quasi-invariance of $\rho_{s,r}$ (and $\mu_{s,r}$) under $\Psi(t)$

$\xrightarrow{r \rightarrow \infty} \implies$ quasi-invariance of μ_s under $\Psi(t)$!!

Nonlinear wave equation: Duhamel part enjoys 1-smoothing:

$$u(t) = S(t)(u_0, u_1) + \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} |u|^{p-1} u(t') dt'$$

Gaussian measure on $(u, \partial_t u)$: $\vec{\mu}_{s+1}(u, \partial_t u) = \mu_{s+1} \otimes \mu_s(u, \partial_t u)$

- $d = 1$: Tzvetkov '15 (implicit in a remark)

Theorem: Oh-Tzvetkov '17 ($d = 2$, defocusing cubic NLW)

Let $s \geq 2$ be an even integer. Then, $\vec{\mu}_{s+1}$ is quasi-invariant under the defocusing cubic NLW on \mathbb{T}^2

- A typical element (u, v) under $\vec{\mu}_{s+1}$ lives in $\mathcal{H}^\sigma = H^\sigma \times H^{\sigma-1}$, $\sigma < s$.
Given a fixed $(h_1, h_2) \in \mathcal{H}^{\sigma+1}$, consider $T_h : (u, v) \mapsto (u, v) + (h_1, h_2)$
Cameron-Martin $\implies \vec{\mu}_{s+1}$ and its transported measure are singular
- Given $(u_0, u_1) \in \mathcal{H}^\sigma$, we only have the nonlinear part for NLW in $\mathcal{H}^{\sigma+1}$

Main difficulty: energy estimate: $\partial_t \|(u, \partial_t u)\|_{\mathcal{H}^{s+1}}^2$

- *renormalized* energy (but **no** renormalization for the equation)

⇐ We achieve this by introducing a *simultaneous* renormalization on both the \mathcal{H}^{s+1} -energy functional and its time derivative

- We establish a renormalized energy estimate in the *probabilistic* setting
- We need to renormalize the energy even if $s \gg 1!!$

In the following, we consider defocusing NLKG (for simplicity):

$$\partial_t^2 u + (1 - \Delta)u = -u^3$$

with Hamiltonian $E(u) = \frac{1}{2} \int (\partial_t u)^2 + \frac{1}{2} \int (Ju)^2 + \frac{1}{4} \int u^4$, $J = \sqrt{1 - \Delta}$

Goal: Define a renormalized energy $E_{s,\infty} \sim \|(u, \partial_t u)\|_{\mathcal{H}^{s+1}}^2$
with a good ∂_t -estimate

Ans: $E_{s,\infty} = \frac{1}{2} \int (J^s \partial_t u)^2 + \frac{1}{2} \int (J^{s+1} u)^2 + \underbrace{\frac{3}{2} \int (J^s u)^2 u^2}_{= \infty, \text{ a.s.}} - \frac{3}{2} \infty \int u^2$

\Leftarrow Both $E_{s,\infty}$ and $\partial_t E_{s,\infty}$ behave “well”

Define σ_N by

$$\sigma_N = \mathbb{E}_{\tilde{\mu}_{s+1}} \left[\int (J^s \mathbf{P}_{\leq N} u)^2 \right] = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \frac{1}{1 + |n|^2} \sim \log N \rightarrow \infty$$

\implies For each $p < \infty$, we have

$$X_N(\omega) := \underbrace{\int (J^s \mathbf{P}_{\leq N} u)^2}_{\rightarrow \text{“}\infty - \infty\text{”}} - \sigma_N = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \frac{|g_n|^2 - 1}{1 + |n|^2} \in L^p(\Omega)$$

with uniform bounds in $N \in \mathbb{N}$.

$\implies X_N$ converges to X_∞ in $L^p(\Omega)$ for any $p < \infty$, allowing us to define

$$X_\infty(\omega) = \int (J^s u)^2 - \sigma_\infty := \lim_{N \rightarrow \infty} \left\{ \int (J^s \mathbf{P}_{\leq N} u)^2 - \sigma_N \right\}$$

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$$\begin{aligned}
\frac{1}{2} \partial_t \|(u, \partial_t u)\|_{\mathcal{H}^{s+1}}^2 &= -3 \int (\partial_t J^s u) J^s u \cdot u^2 + \text{l.o.t.} \\
&\stackrel{\text{IBP}}{=} -\frac{3}{2} \partial_t \left[\int (J^s u)^2 u^2 \right] + 3 \int (J^s u)^2 \partial_t u \cdot u + \text{l.o.t.} \\
&= -\frac{3}{2} \partial_t \left[\int \mathbf{P}_{\neq 0} [(J^s u)^2] \cdot \mathbf{P}_{\neq 0} [u^2] \right] + 3 \int \mathbf{P}_{\neq 0} [(J^s u)^2] \cdot \mathbf{P}_{\neq 0} [\partial_t u \cdot u] + \text{l.o.t.} \\
&\quad - \underbrace{\frac{3}{2} \partial_t \left[\int (J^s u)^2 \int u^2 \right]}_{=\infty} + \underbrace{3 \int (J^s u)^2 \int \partial_t u \cdot u}_{=\infty}
\end{aligned}$$

With σ_N , we have

$$\begin{aligned}
&-\frac{3}{2} \partial_t \left[\int (J^s u)^2 \int u^2 \right] + 3 \int (J^s u)^2 \int \partial_t u \cdot u \\
&= -\frac{3}{2} \partial_t \left[\underbrace{\left(\int (J^s u)^2 - \sigma_N \right)}_{=X_N} \int u^2 \right] + 3 \underbrace{\left(\int (J^s u)^2 - \sigma_N \right)}_{=X_N} \int \partial_t u \cdot u.
\end{aligned}$$

Define the **renormalized energy** $E_{s,N}(u, \partial_t u)$ by

$$\begin{aligned} E_{s,N}(u, \partial_t u) &= \frac{1}{2} \int (J^s \partial_t u)^2 + \frac{1}{2} \int (J^{s+1} u)^2 + \frac{3}{2} \int (J^s u)^2 u^2 - \frac{3}{2} \sigma_N \int u^2 \\ &= \frac{1}{2} \int (J^s \partial_t u)^2 + \frac{1}{2} \int (J^{s+1} u)^2 + \frac{3}{2} \int \mathbf{P}_{\neq 0}[(J^s u)^2] \cdot \mathbf{P}_{\neq 0}[u^2] \\ &\quad + \frac{3}{2} \left(\int (J^s u)^2 - \sigma_N \right) \int u^2 \end{aligned}$$

$$\implies \partial_t E_{s,N}(u) = 3 \int \mathbf{P}_{\neq 0}[(J^s u)^2] \cdot \mathbf{P}_{\neq 0}[\partial_t u \cdot u] + 3 \left(\int (J^s u)^2 - \sigma_N \right) \int \partial_t u \cdot u + \text{l.o.t.}$$

Probabilistic renormalized energy estimate:

$$\left\{ \int_{\{E(\mathbf{P}_{\leq N} u, \mathbf{P}_{\leq N} v) \leq r\}} \left| \partial_t E_{s,N}(\pi_N \Phi_N(t)(u, v)) \Big|_{t=0} \right|^p d\mu_s(u, v) \right\}^{\frac{1}{p}} \lesssim p$$

$$\implies E_{s,N} \rightarrow E_{s,\infty} = \frac{1}{2} \int (J^s \partial_t u)^2 + \frac{1}{2} \int (J^{s+1} u)^2 + \frac{3}{2} \int (J^s u)^2 u^2 - \frac{3}{2} \infty \int u^2, \text{ a.s.}$$

and $E_{s,\infty}$ satisfies the same ∂_t -bound

Remarks on Chapter 3

- Propagation of the $W^{\sigma,r}$ -regularity, $\sigma < s - \frac{d}{2}$ and $r \leq \infty$
- We showed mutual absolute continuity of the transported measure $\Phi(t)_* \mu_s$ and the original Gaussian measure μ_s . Our argument, however, does not tell us much about the *time-dependent* Radon-Nikodym derivative (in $L^1(\mu_s)$) of $\Phi(t)_* \mu_s$ with respect to μ_s . It would be interesting to study more about the resulting Radon-Nikodym derivatives
 - higher integrability in $L^p(\mu_s)$, $p > 1$?
 - compactness in time? property of its time average?
- By an argument analogous to that for invariant measure, we can obtain

$$\|u(t)\|_{H^\sigma} \lesssim C(u_0^\omega)(1 + |t|)^{\alpha(s)} \text{ for any } t \in \mathbb{R},$$

where $\alpha(s) \rightarrow \infty$, as $s \rightarrow \infty$. It is very far from the logarithmic bound for invariant measures and may be obtained by deterministic techniques.

Q: Can we establish *quantitative* versions of quasi-invariance and prove new growth bounds on higher Sobolev norms of solutions in a probabilistic manner?

- Our current understanding of the corresponding question for the (more complicated) NLS is very poor (except for 1- d cubic NLS)...

Remarks & comments on Chapters 1 and 2

$\rho =$ invariant (Gibbs) measure

Dynamical properties?

① Recurrence property: Poincaré, Furstenberg '77

② ρ invariant $\implies u(t) \stackrel{\mathcal{D}}{\sim} u(0)$ but how are $u(t)$ and $u(0)$ related?

Can we say anything about the space-time covariance $\mathbb{E}_\rho[u(x, t)\overline{u(y, 0)}]$?

- Lukkarinen-Spohn '11: weakly nonlinear & large box limit of lattice NLS

③ Ergodicity and 'asymptotic stability' of ρ ?

- Completely open
- These questions have been answered for some stochastic PDEs. This is mainly due to *uniqueness* of invariant measures. However, for Hamiltonian PDEs, there are more than one (formally) invariant measures and such questions are out of reach at this point...

This recent development also led to

- ① **Probabilistic well-posedness** beyond deterministic analysis:
 - almost sure well-posedness with *rough & random* initial data
 - Global (*without* invariant measures): Colliander-Oh '12, Nahmod-Pavlović-Staffilani '13, Burq-Tzvetkov '14, Lührmann-Mendelson '14, Bényi-Oh-Pocovnicu '15 (conditional), Pocovnicu '15, Oh-Pocovnicu '16, Killip-Murphy-Viřan '17, Oh-Okamoto-Pocovnicu '17
- ② **Singular stochastic dispersive PDEs:** *space-time white noise forcing*
 - stochastic KdV on \mathbb{T} : LWP (Oh '09), global dynamics (Oh-Quastel-Sosoe '17)
 - stochastic NLW on \mathbb{T}^2 : LWP (Gubinelli-Koch-Oh '17)
GWP (Gubinelli-Koch-Oh-Tolomeo '17)
time-dependent renormalization
 - stochastic cubic NLS on \mathbb{T} : *completely open*
important in fiber optics

- Gibbs measure on \mathbb{T}^3 is renormalizable only for $p = 3$ & defocusing
 - Wick ordering is not enough (need second order correction)
 - very rough $\sim H^{-\frac{1}{2}-}(\mathbb{T}^3)$

- Stochastic quantization equation:

$$\partial_t u = \Delta u - u^3 + \infty \cdot u + \underbrace{\xi}_{\text{space-time white noise}}$$

- formally preserves the Gibbs measure
 - “local well-posedness”: Hairer '14 (regularity structure), Kupiainen '16 (RG method), Catellier-Chouk '16 (paracontrolled distribution introduced by Gubinelli-Imkeller-Perkowski '15)
 - invariance of Gibbs measure: Hairer-Matetski '15
 - global well-posedness: Mourrat-Weber '16
- (renormalized) defocusing cubic NLS/NLW on \mathbb{T}^3 ?

Completely open