Coproducts of Operads, and the W-Construction

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The point of this document is to describe coproducts in the category of operads, and to observe that the functor $P \longmapsto P + \widetilde{I}$ closely resembles the W-construction, where $+$ denotes coproduct and $\widetilde{I}$ is a certain operad.

1 Terminology

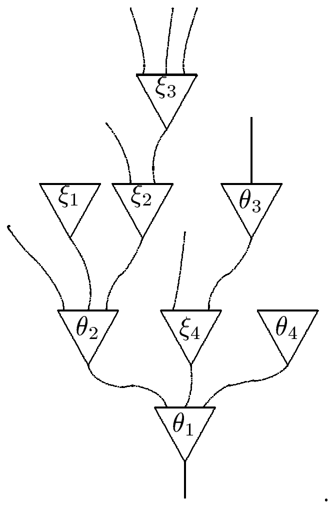
'Operad' will mean single-coloured, non-symmetric operad of sets. I haven't thought about what happens if we have topologies, symmetries or several colours. The unit element of an operad P will be written $\iota_P \in P(1)$.

2 Description of the Coproduct

Let P and Q be operads. We describe their coproduct $P + Q$ in the category of operads in three steps.

i. For $n \geq 0$, let $D(n)$ be the set of trees with n leaves (= twigs, = tails), with vertices labelled by elements of P or of Q of the appropriate arities. For instance, the following is an element of $D(5)$, where $\theta_1, \theta_2 \in P(3), \theta_3 \in$

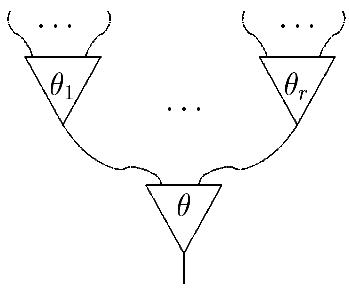
$P(1), \theta_4 \in P(0), \xi_1 \in Q(0), \xi_2 \in Q(2), \xi_3 \in Q(3), \xi_4 \in Q(2)$:



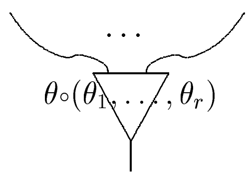
Formally, D is the free operad on the object $(P(n) + Q(n))_{n \in \mathbb{N}}$ of the category $\mathbf{Set}^{\mathbb{N}}$.

ii. We next define an equivalence relation $\sim$ on $D(n)$, for each n. This equivalence relation is generated by the following rules:

(a) whenever

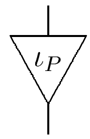


$(\theta \in P(r), \theta_1 \in P(k_1), \ldots, \theta_r \in P(k_r))$ appears as a subtree of a diagram, then this subtree may equivalently be replaced by



(b) whenever

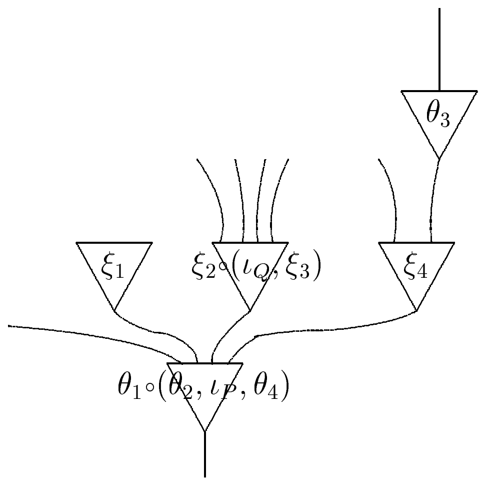
appears in a diagram, it may be replaced by



(c) as in (iia), but with Q in place of P

(d) as in (iib), but with Q in place of P.

For instance, this tells us that the element of $D(5)$ drawn above is equivalent to the following element of $D(5)$:



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iii. There is a natural operad structure on the sequence of sets $(D(n)/\sim)_{n\in\mathbb{N}}$, and one can see that this is the coproduct of the operads P and Q.

3 Monoids and Operads

There is a functor

$$\widetilde{(\ )} : \text{(monoids)} \longrightarrow \text{(operads)}$$

sending a monoid M to the operad $\widetilde{M}$ defined by $\widetilde{M}(1) = M$, $\widetilde{M}(n) = \emptyset$ for $n \neq 1$, and with multiplication and unit as in M. (This functor is, in fact, left adjoint to the functor sending an operad P to the monoid $P(1)$.)

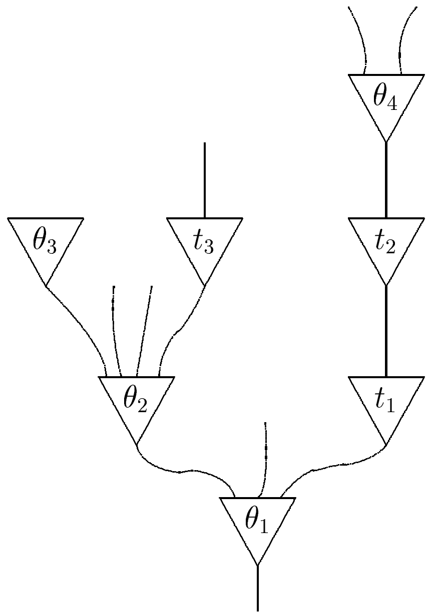
In particular, consider the monoid $(I, *, 0)$, where I is the unit interval $[0,1]$ and

$$t_1 * t_2 = t_1 + t_2 - t_1 t_2$$

(or $t_1 * t_2 = \max\{t_1, t_2\}$, if preferred). This gives rise to an operad $\widetilde{I}$.

4 The Functor $(\text{—}) + \widetilde{I}$

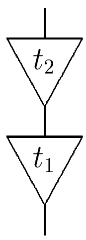
The description in section 2 of the coproduct of a pair of operads gives, in particular, a description of what $P + \widetilde{I}$ is for an operad P. So, we have $(P + \widetilde{I})(n) = D(n)/\sim$, where a typical member of $D(4)$ is a diagram



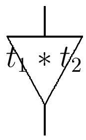
$(*)$

$(\theta_1 \in P(3), \theta_2 \in P(4), \theta_3 \in P(0), \theta_4 \in P(2), t_1, t_2, t_3 \in [0,1])$, and $\sim$ is generated by the relations (iia) and (iib) above, together with the relations

- whenever

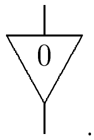


$(t_1, t_2 \in [0,1])$ appears in a diagram, it may equivalently be replaced by

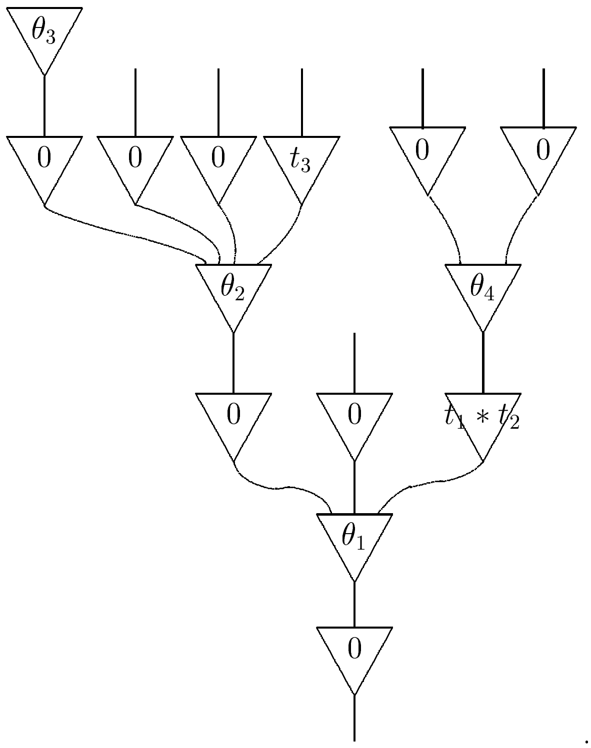


- whenever

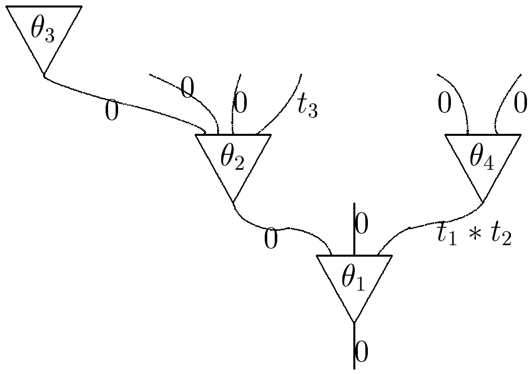
appears in a diagram, it may equivalently be replaced by



For instance, the element of $D(4)$ depicted in $(*)$ is equivalent to



Evidently, this diagram could also be represented as a tree of elements of P in which the edges are labelled with numbers in $[0,1]$:



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So the operad $P+\widetilde{I}$ is almost exactly the operad $W(P)$ defined by the Boardman-Vogt method. As far as I can see, the only point of difference is that in Boardman-Vogt, 'by convention, the roots and twigs have length 1' (*Homotopy Invariant Algebraic Structures* ..., p. 73), whereas in the coproduct they have length 0. (The element 1 of the monoid I plays no special role; the unit element is 0.)

Incidentally, in Remark 3.2 of *ibid* it is noted that in the definition of $W(P)$, the monoid I could be replaced by some other monoid. We have seen that more generally, the monoid I could be replaced by some other *operad* (where a monoid is viewed as a special kind of operad via $\widetilde{(\ )}$). In other words, we have described $W(P)$ as (almost exactly) $P+Q$ for a certain operad Q.

5 Augmentation

The augmentation map $W(P) \longrightarrow P$ (for an operad P) is defined by forgetting edge-lengths. Here we see how this map arises from the coproduct description.

Let 1 be the one-element monoid. The unique monoid map $! : I \longrightarrow 1$ induces an operad map $\widetilde{!} : \widetilde{I} \longrightarrow \widetilde{1}$. But $\widetilde{1}$ is the initial operad, so $P+\widetilde{1} \cong P$ for any operad P. We therefore have a natural map, 'augmentation',

$$P+\widetilde{I} \xrightarrow{1_P+\widetilde{!}} P+\widetilde{1} \cong P.$$