Large sets

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Reference: Large sets 1–13, *n-Category Café*, 2021

Purpose of this talk

Provide some evidence for the hypothesis:

> Hypothesis Everything in traditional, membership-based set theory that's relevant to the rest of mathematics can be done smoothly in categorical set theory.

Do this by working out the beginning of the theory of large cardinals in Lawvere's Elementary Theory of the Category of Sets (ETCS).

Plan

1. Introduction to ETCS
2. Large sets in ETCS

1. Introduction to ETCS

Lawvere, An elementary theory of the category of sets, 1964

Lawvere and Rosebrugh, *Sets for Mathematics*, 2003

Leinster, Rethinking set theory, 2012

ETCS in one sentence

ETCS is the following theory:

> Sets and functions form a well-pointed topos with natural numbers and choice.

That's the high-tech way to say it. But crucially:

To state ETCS, we do not need the general notion of category.

Compare: you can discuss addition and multiplication of integers without the general notion of ring.

ZFC and ETCS

	ZFC	ETCS
Are elements of a set also sets?	always	never
Given sets X and Y, can you ask whether $X \in Y$?	yes	no
Does '$X \cap Y$' make sense for arbitrary X and Y?	yes	no
Is everything isomorphism-invariant?	no	yes
Sets	primitive	primitive
$\in$	primitive	derived
Functions	derived	primitive
Composition	derived	primitive

Cardinals and ordinals

In traditional, membership-based set theory:

- An ordinal is cleverly defined as a set with certain properties. Every well-ordered set is order-isomorphic to a unique ordinal.
- A cardinal is defined as an ordinal with certain properties. Every set is in bijection with a unique cardinal.

In categorical set theory, everything is isomorphism-invariant. So:

- No need to talk about ordinals. Just talk about well-ordered sets.
- No need to talk about cardinals. Just talk about sets.

Hence ‘large sets’, not ‘large cardinals’.

ETCS in elementary terms

ETCS takes as its starting data:

- some things called sets;
- for each set X and set Y, some things called functions from X to Y, written as $f\colon X \longrightarrow Y$;
- for each set X, set Y and set Z, an operation of composition, assigning to each $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ a function $gf\colon X \longrightarrow Z$.

The axioms are as follows. . .

ETCS in elementary terms

Informally stated, the axioms are:

1. Composition of functions is associative and has identities.
2. There is a set with exactly one element.
   Formally: there exists a terminal set, 1.
   An element of a set X is a function $x\colon 1 \longrightarrow X$; then write $x \in X$.
3. There is a set with no elements.
4. A function is determined by its effect on elements.
   That is: if $f, g\colon X \longrightarrow Y$ and $fx = gx$ for all $x \in X$ then $f = g$.
5. Given sets X and Y, one can form their cartesian product $X \times Y$.
6. Given sets X and Y, one can form the set of functions from X to Y.
7. Given $f\colon X \longrightarrow Y$ and $y \in Y$, one can form the inverse image $f^{-1}(y)$.
8. The subsets of a set X correspond to the functions from X to $\{0, 1\}$.
   Or really: there is a set Ω such that injections into a set X correspond to functions $X \longrightarrow \Omega$. The axioms imply that Ω has exactly two elements.
9. The natural numbers form a set.
10. Every surjection has a right inverse.

Families of sets

Let I be a set.

A family of sets indexed by I is a set X together with a function $p \colon X \longrightarrow I$.

Write $X_i = p^{-1}(i)$, and think of it as the ith member of the family.

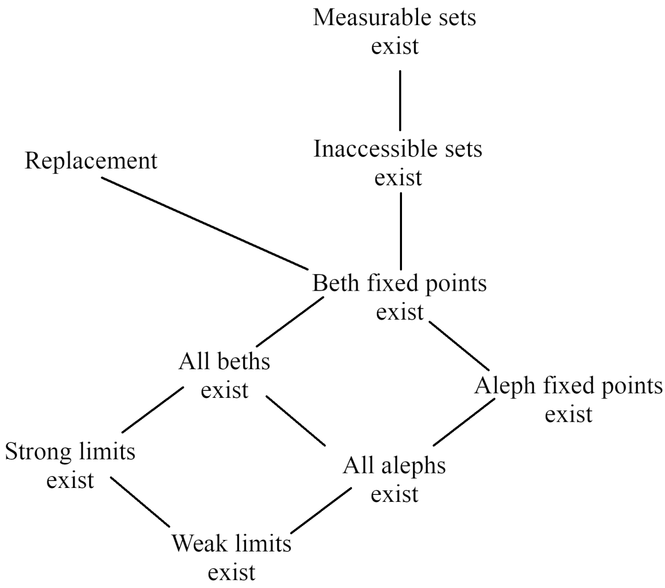
For sets A and B, write $A \leq B$ if there exists an injection $A \longrightarrow B$.

Theorem *For any family $X \longrightarrow I$ with I nonempty, there is some $i \in I$ such that $X_i \leq X_j$ for all j.*

Roughly: sets are well-ordered by $\leq$.

2. Large sets in ETCS

The large set conditions we'll consider



Strong limits

An infinite set X is a strong limit if for all Y,

$$Y < X \Rightarrow 2^Y < X.$$

E.g. $\mathbb{N}$ is a strong limit.

Theorem *A set X is an uncountable strong limit $\iff$ the sets $< X$ are a model of ETCS.*

Corollary *It is consistent with ETCS that there are no uncountable strong limits.*

Proof Take a model of ETCS.

Call a set 'small' if it's $<$ every uncountable strong limit.

Then the small sets are a model of ETCS containing no uncountable strong limits.

Weak limits

There are two standard ways of making a set X bigger:

- take the power set 2^X;
- take the successor X^+ (the smallest set $> X$).

The generalized continuum hypothesis says that $X^+ \cong 2^X$ for all infinite X.

An infinite set X is a weak limit if

$$Y < X \Rightarrow Y^+ < X,$$

or equivalently if X is not a successor.

The generalized continuum hypothesis is consistent with ETCS, so:

Corollary *It is consistent with ETCS that there are no uncountable weak limits.*

Well-ordered sets

An ordered set is well-ordered if every nonempty subset has a least element.

For WO sets W and W', write $W \preceq W'$ if W is isomorphic to a downwards closed subset of W'.

Then $\preceq$ is a well-order (up to isomorphism) on the class of WO sets.

Sets versus well-ordered sets

Given a well-ordered set W, there is an underlying set $U(W)$.

Given a set X, let $I(X)$ denote X equipped with a $\preceq$-least (initial) well-order.

This defines an adjunction

$$(\mathbf{WOSet}, \preceq) \underset{I}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} (\mathbf{Set}, \leq)$$

with $U \circ I \simeq \mathrm{id}_{\mathbf{Set}}$.

It restricts to an equivalence

$$(\text{initial WO sets}, \preceq) \; \simeq \; (\mathbf{Set}, \leq).$$

The index of a set

For an infinite set X, define a well-ordered set

$$\begin{aligned}\mathrm{Index}(X) &= \{\text{iso classes of infinite sets } < X\} \\ &= \{A \in 2^X \;:\; \mathbb{N} \leq A < X\}/\cong,\end{aligned}$$

ordered by $\leq$.

E.g. $\mathrm{Index}(\mathbb{N}^{++})$ consists of the iso classes of $\mathbb{N}$ and $\mathbb{N}^+$, so $\mathrm{Index}(\mathbb{N}^+) \cong \{0, 1\}$.

The Index construction defines an order-embedding

$$\mathrm{Index}\colon (\text{infinite sets}, \leq) \hookrightarrow (\mathbf{WOSet}, \preceq).$$

If $\mathrm{Index}(X) \cong \omega$ then X is a weak limit. So:

Corollary *It is consistent with ETCS that there is no set with index ω.*

Alephs

Recall that we have an order-embedding

$$\mathrm{Index}\colon (\text{infinite sets}, \leq) \hookrightarrow (\mathbf{WOSet}, \preceq).$$

Let W be a WO set. If there is some set X such that $\mathrm{Index}(X) \cong W$, we write X as $\aleph_W$ and say that $\aleph_W$ exists.

Equivalent recursive definition $\aleph_W$ exists if there is some infinite set $> \aleph_V$ for every $V \prec W$, and in that case, $\aleph_W$ is the smallest such set.

It is consistent with ETCS that $\aleph_\omega$ does not exist.

Fact All alephs exist $\iff$ for every set I, there exists a family $X \longrightarrow I$ with $X_i \ncong X_j$ whenever $i \neq j$.

Beths

Thought For sets X and Y, we have $X < Y \iff X^+ \leq Y$.

Replacing X^+ by 2^X, we might ask whether $2^X \leq Y$.

Recursive definition $\beth_W$ exists if there is some infinite set $\geq 2^{\beth_V}$ for every $V \prec W$, and in that case, $\beth_W$ is the smallest such set.

E.g. $\beth_0 = \mathbb{N}$, $\beth_1 = 2^{\mathbb{N}}$, $\beth_2 = 2^{2^{\mathbb{N}}}$, ..., and $\beth_\omega$ (if it exists) is their supremum.

Fact All beths exist $\iff$ for every set I, there exists a family $X \longrightarrow I$ with $2^{X_i} \leq X_j$ or $2^{X_j} \leq X_i$ whenever $i \neq j$.

The axiom 'all beths exist' is very powerful!

E.g. it implies $\mathcal{P}^W(X)$ exists for all sets X and WO sets W, and is enough to prove the Borel determinacy theorem.

Beth fixed points

Consider the assignments

$$\text{set } X \;\mapsto\; \text{WO set } I(X) \;\mapsto\; \text{set } \beth_{I(X)}.$$

E.g.

$$3 \mapsto I(3) = \{0, 1, 2\} \mapsto \beth_3 = 2^{2^{2^{\mathbb{N}}}}.$$

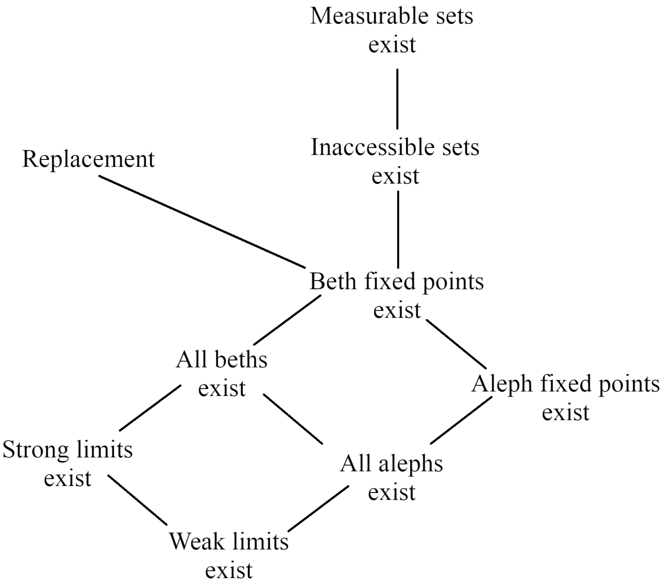
Can show that $\beth_{I(X)} \geq X$ for all X.

X is a beth fixed point if $\beth_{I(X)} \cong X$.

Any beth fixed point is an uncountable strong limit.

Theorem *X is a beth fixed point $\iff$ the sets $< X$ are a model of ETCS + all beths exist.*

Corollary *It is consistent with ETCS + all beths exist that there are no beth fixed points.*


Measurable sets exist
Replacement
Inaccessible sets exist
Beth fixed points exist
All beths exist
Aleph fixed points exist
Strong limits exist
All alephs exist
Weak limits exist

Inaccessible sets

An infinite set X is regular if for every family $S \longrightarrow I$ of sets, if $I < X$ and each $S_i < X$ then $S < X$.

(That is: a disjoint union of $< X$ sets, each $< X$, is $< X$.)

E.g. $\mathbb{N}$ is regular, as a finite union of finite sets is finite.

A set is inaccessible if it is uncountable, regular, and a strong limit.

Any inaccessible set is a beth fixed point. In fact:

Theorem *If X is inaccessible then there are unboundedly many beth fixed points $< X$.*

Corollary *It is consistent with ETCS + (there are unboundedly many beth fixed points) that there are no inaccessibles.*

Measurable sets

Given a set X, can look for $\{0,1\}$-valued probability measures on X, defined on every subset of X.

E.g. For each $x \in X$, there is the trivial measure (Dirac delta) δ_x, where $\delta_x(A) = 1$ iff $x \in A$.

Can ask that the measure is not just *countably* additive, but 'Y-fold additive' for all $Y < X$.

A set X is measurable if it is uncountable and admits a nontrivial $\{0,1\}$-valued probability measure that is Y-fold additive for all $Y < X$.

Measurable sets

Every measurable set is inaccessible. In fact:

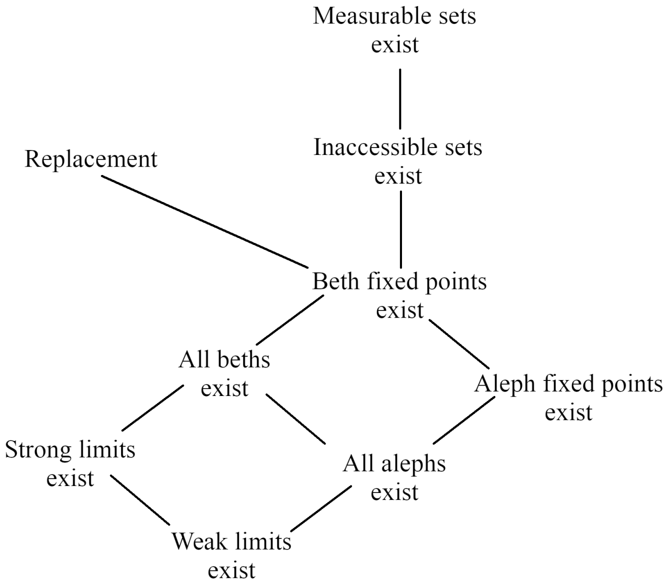
Theorem *For every measurable set X, there are unboundedly many inaccessible sets $< X$.*

This is much harder than all previous theorems.

Corollary *It is consistent with ETCS + (there exist unboundedly many inaccessibles) that there are no measurable sets.*

Theorem (Isbell, 1960) *There are no measurable sets in* **Set** *iff the countable sets are codense in* **Set**.

'Codense' means that every set is canonically a limit of countable sets.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Replacement

(McLarty, Exploring categorical structuralism, 2004)

The axiom scheme of replacement (R) states (slightly informally):

> Take a set I and a first-order formula that for each $i \in I$ specifies a set $F(i)$, uniquely up to isomorphism. Then there exists a function into I with fibres $F(i)$.

It is equivalent to transfinite recursion, and implies that all beths exist, that there are unboundedly many beth fixed points, and much more.

But it is consistent with ETCS+R that there are no inaccessible sets.

Theorem *ETCS+R is bi-interpretable with ZFC.*

'Bi-interpretable' means equivalent in the strongest possible sense.

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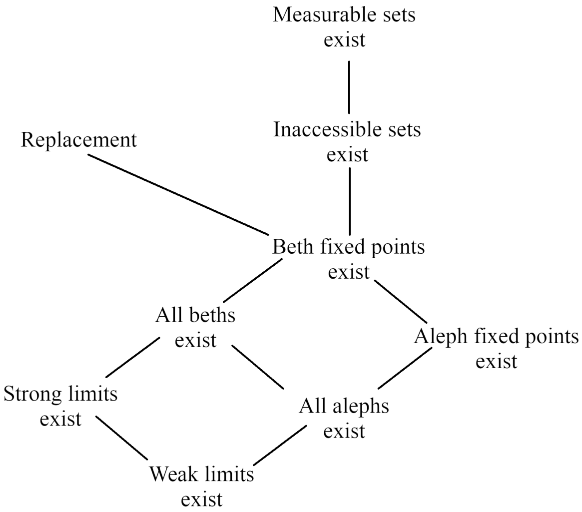

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