General Topology 1

Metric and topological spaces

The deadline for handing this work in is **1pm on Monday 29 September 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

All the questions will be assessed except where noted otherwise. Please take care over communication and presentation. Your answers should be coherent, logical arguments written in full sentences. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. Let $b \geq 2$ be an integer.
   (i) For distinct integers x and y, define $N(x, y)$ to be the largest integer n such that b^n divides $x - y$, and put $d(x, y) = b^{-N(x,y)}$. For integers x, put $d(x, x) = 0$. Prove that d is a metric on $\mathbb{Z}$.
   (ii) Give a nontrivial example of a convergent sequence in this metric.

2. *In the second lecture, some of you discovered a new metric space: take the Euclidean metric on $\mathbb{R}^n$, then round distances up to the next integer. This question extends this discovery to arbitrary metric spaces.*

   Recall that the **ceiling** $\lceil a \rceil$ of a real number a is the smallest integer greater than or equal to a.

   (i) Show that $\lceil a \rceil + \lceil b \rceil \geq \lceil a + b \rceil$ for all $a, b \in \mathbb{R}$.
   (ii) Let (X, d) be a metric space. Define a function $\lceil d \rceil : X \times X \to [0, \infty)$ by $\lceil d \rceil(x, y) = \lceil d(x, y) \rceil$ $(x, y \in X)$. Using part (i), or otherwise, show that $\lceil d \rceil$ is a metric on X.
   (iii) Is $\lceil d \rceil$ topologically equivalent to d? Give a proof or counterexample.

3. Prove the inequalities in Example A3.4(ii): that for the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$,
   $$d_\infty(x, y) \leq d_1(x, y) \leq n d_\infty(x, y), \qquad d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$$
   $(x, y \in \mathbb{R}^n)$. Deduce that d_1, d_2 and d_∞ are topologically equivalent.

4. Let (X, d) be a metric space. Define a function $d' : X \times X \to [0, \infty)$ by
   $$d'(x, y) = \min\{d(x, y), 1\}$$
   $(x, y \in X)$. Prove that d' is a metric, and that d' is topologically equivalent to d.

   The number 1 could be changed to any other positive constant here. The fact that d' is topologically equivalent to d can be understood as follows. When we define 'open set' in a metric space, it is only the small *distances that matter. So, if we modify d in a way that keeps all the small distances the same, the induced topology is unchanged.*

5. Let X be a set. A subset U of X is **cofinite** if $X \setminus U$ is finite. Prove that there is a topology on X consisting of the cofinite subsets together with $\emptyset$. (This is called the **cofinite topology**.)

   Assume now that the set X is infinite. Is X with the cofinite topology T_1? Hausdorff? Metrizable?

6. Let k be a field and $n \geq 0$. Denote by $k[X_1, \ldots, X_n]$ the set of all polynomials over k in commuting variables $X_1, \ldots, X_n$. ('Commuting' means that $X_1 X_2 = X_2 X_1$ etc.) For $S \subseteq k[X_1, \ldots, X_n]$, put
   $$V(S) = \{(x_1, \ldots, x_n) \in k^n : p(x_1, \ldots, x_n) = 0 \text{ for all } p \in S\}.$$
   A subset V of k^n is **Zariski closed** if $V = V(S)$ for some $S \subseteq k[X_1, \ldots, X_n]$, and **Zariski open** if its complement is Zariski closed.

(i) Prove that the Zariski open subsets of k^n define a topology on k^n.

(ii) Prove that the Zariski topology need not be Hausdorff. (That is, show that for at least one choice of k and n, the topology on k^n defined in (i) is not Hausdorff.)

Challenge questions

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7. Prove that every metric space is normal.

8. *In the very first lecture of the course, metric spaces were motivated by examples such as 'distance on foot within a city'. Essentially this means that we are taking a subset X of $\mathbb{R}^n$ (in this case, the parts of the city where one can walk) and defining the distance between two points in X to be the length of the shortest path in X between them. Here we make this idea precise.*

   A map $f\colon X \to Y$ of metric spaces is **distance-decreasing** if $d(f(x), f(x')) \leq d(x, x')$ for all $x, x' \in X$.

   Let $X = (X, d)$ be a metric space. Suppose that for all $x, y \in X$, there exist a real number $D \geq 0$ and a distance-decreasing map $\gamma : [0, D] \to X$ such that $\gamma(0) = x$ and $\gamma(D) = y$. *('It is possible to walk from any point to any other'.)*

   Define a function $\hat{d} : X \times X \to [0, \infty)$ by

$$\hat{d}(x, y) = \inf\{D \in [0, \infty) : \text{ there exists a distance-decreasing map } \gamma : [0, D] \to X \text{ such that } \gamma(0) = x \text{ and } \gamma(D) = y\}$$

   $(x, y \in X)$. Prove that $\hat{d}$ is a metric on X, and that $\hat{\hat{d}} = \hat{d}$.

General Topology 2

Topological spaces and continuous maps

The deadline for handing this work in is **1pm on Monday 13 October 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

All the questions will be assessed except where noted otherwise. Please take care over communication and presentation. Your answers should be coherent, logical arguments written in full sentences. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. Let U be a subset of a metric space X. Prove that U is open in X if and only if U can be expressed as a union of open balls in X.

2. Let X and Y be topological spaces, with Y Hausdorff. Let $f, g \colon X \to Y$ be continuous maps. Prove that $\{x \in X : f(x) = g(x)\}$ is closed in X.

3. Find an example of a continuous bijection that is not a homeomorphism, different from the examples in the notes.

4. *In this question, you will prove that the n-sphere with a point removed is homeomorphic to $\mathbb{R}^n$.*

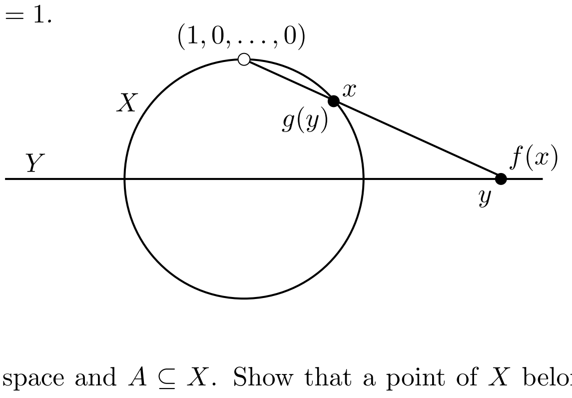
   Let $n \geq 1$, and put $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$. Put $X = S^n \setminus \{(1, 0, \ldots, 0)\}$ and $Y = \{(y_0, \ldots, y_n) \in \mathbb{R}^{n+1} : y_0 = 0\}$, both with the Euclidean metric. Thus, X is the n-sphere with a point removed and $Y \cong \mathbb{R}^n$.

   (i) For $x = (x_0, \ldots, x_n) \in X$, let $f(x)$ be the unique point of Y such that $(1, 0, \ldots, 0)$, x and $f(x)$ are collinear. Find an explicit formula for $f(x)$. *(Hint: collinearity means that $f(x) = \lambda(x)(1, 0, \ldots, 0) + (1 - \lambda(x))x$ for some $\lambda(x) \in \mathbb{R}$. Find $\lambda(x)$.)*

   (ii) For $y = (y_0, \ldots, y_n) \in Y$, let $g(y)$ be the unique point of X such that $(1, 0, \ldots, 0)$, y and $g(y)$ are collinear. Find an explicit formula for $g(y)$. *(Hint: use a similar method to (i).)*

   (iii) Prove that $f \colon X \to Y$ and $g \colon Y \to X$ are inverse to each other, and deduce that X is homeomorphic to Y.

   The map f is called **stereographic projection** *from the punctured n-sphere to $\mathbb{R}^n$. It is shown here for $n = 1$.*



5. Let X be a metric space and $A \subseteq X$. Show that a point of X belongs to $\mathrm{Cl}(A)$ if and only if it is the limit of some sequence of elements of A.

6. Let X be a topological space. Prove that $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$ for all $A, B \subseteq X$, but that this identity need not hold when unions are replaced by intersections, nor for infinite unions.

Challenge questions

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7. A topological space is said to be **second countable** if it has a countable basis. Prove that $\mathbb{R}^n$ is second countable.

8. Let X be a topological space, and write $\mathscr{P}(X)$ for the set of subsets of X. We have functions $\mathrm{Cl}, \mathrm{Int}\colon \mathscr{P}(X) \to \mathscr{P}(X)$. By composing them (perhaps several times over), we can build further functions $\mathscr{P}(X) \to \mathscr{P}(X)$, such as $\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Int}$ and $\mathrm{Int}\,\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}$.

   Evidently these functions are not *all* the same: e.g. Cl is not the same as Int, and $\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}$ is not the same as Cl (by the example in Warning A8.17). But we also know that some *are* the same: e.g. Lemma A8.3(ii) states that $\mathrm{Cl}\,\mathrm{Cl} = \mathrm{Cl}$.

   Determine which are the same and which are different. Hint: although in principle there could be infinitely many different functions $\mathscr{P}(X) \to \mathscr{P}(X)$ that can be obtained by repeatedly taking closures and interiors, there are in fact only finitely many.

General Topology 3

Subspaces, products and quotients

The deadline for handing this work in is **1pm on Monday 27 October 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

All the questions will be assessed except where noted otherwise. Please take care over communication and presentation. Your answers should be coherent, logical arguments written in full sentences. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. (i) Prove that:
   a. a subspace of a discrete space is always discrete;
   b. a product of two discrete spaces is always discrete;
   c. a quotient of a discrete space is always discrete.

   (ii) Say whether each of these is true when 'discrete' is replaced by 'indiscrete' throughout, giving proofs or counterexamples.

2. Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$.

   (i) Prove that $\mathrm{Int}(A \times B) = \mathrm{Int}(A) \times \mathrm{Int}(B)$.

   (ii) Prove that $\mathrm{Cl}(A \times B) = \mathrm{Cl}(A) \times \mathrm{Cl}(B)$.

   (iii) When Z is a topological space and $C \subseteq Z$, the **boundary** of C is defined as $\partial C = \mathrm{Cl}(C) \setminus \mathrm{Int}(C)$. Prove that if A is closed in X and B is closed in Y then

   $$\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B).$$

   What does this equation remind you of?

3. Let (X, d_X) and (Y, d_Y) be metric spaces. Define metrics d_1, d_2 and d_∞ on $X \times Y$ by

   $$\begin{aligned} d_1((x,y),(x',y')) &= d_X(x,x') + d_Y(y,y'), \\ d_2((x,y),(x',y')) &= \left(d_X(x,x')^2 + d_Y(y,y')^2\right)^{1/2}, \\ d_\infty((x,y),(x',y')) &= \max\{d_X(x,x'), d_Y(y,y')\}. \end{aligned}$$

   (You're not asked to show that these *are* metrics.) Prove that d_1, d_2 and d_∞ all induce the product topology on $X \times Y$.

   (Hint: before plunging into calculations, think about strategy. You may be able to use work we've already done to make this easier.)

4. For this question, recall that given any set I (perhaps infinite) and any family $(X_i)_{i \in I}$ of sets, there is a product set $\prod_{i \in I} X_i$. Its elements are families $(x_i)_{i \in I}$ with $x_i \in X_i$ for each $i \in I$. For any set Z, a function $f \colon Z \to \prod X_i$ amounts to a function $f_i \colon Z \to X_i$ for each $i \in I$. These f_i are called the 'components' of f. (This is much like Remark A10.6.)

   Let I be a set and let $(X_i)_{i \in I}$ be a family of topological spaces. Find a topology on the set $\prod_{i \in I} X_i$ with the following property: for any space Z and function $f \colon Z \to \prod X_i$, the function f is continuous if and only if each of its components f_i ($i \in I$) is continuous.

   (This is just like Proposition A10.9, but for a potentially infinite number of spaces. There's exactly one topology on $\prod X_i$ with the property required. Warning: it's probably not the topology you'll first guess.)

5. Write down two definitions of compactness for metric spaces. Does either definition generalize easily to topological spaces? What would you guess (without looking it up) is the definition of compactness for topological spaces?

Challenge questions

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6. Prove that the subspace, product and quotient topologies are the *only* topologies that make Propositions A9.10, A10.9 and A11.7 true.

   (For instance, this means the following for subspaces. Let X be a topological space and $A \subseteq X$. Let $\mathcal{T}$ be a topology on A such that Proposition A9.10 holds with respect to $\mathcal{T}$. Then $\mathcal{T}$ is the subspace topology.)

7. Let X be a compact metric space and let $f\colon X \to X$ be an isometry (that is, $d(f(x), f(x')) = d(x, x')$ for all $x, x' \in X$). Prove that f is surjective. *(Hard!)*

8. Prove that if $\mathbb{R}^m \cong \mathbb{R}^n$ then $m = n$.

   (Really seriously hard. If you can find a simple proof, worldwide celebrity will be yours.)

General Topology 4

Compactness

The deadline for handing this work in is **1pm on Monday 10 November 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

All the questions will be assessed except where noted otherwise. Please take care over communication and presentation. Your answers should be coherent, logical arguments written in full sentences. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. True or false? (You do not have to justify your answers.)
   (i) No open cover of a non-compact space has a finite subcover.
   (ii) Every non-compact space has an open cover of which all subcovers are infinite.
   (iii) Every compact subspace of a metrizable space is closed.
   (iv) Every subspace of a discrete space is compact.
   (v) Every product of a finite space with a compact space is compact.
   (vi) Every closed bounded subset of a metrizable space is compact.
   (vii) Every quotient of a compact Hausdorff space is compact Hausdorff.

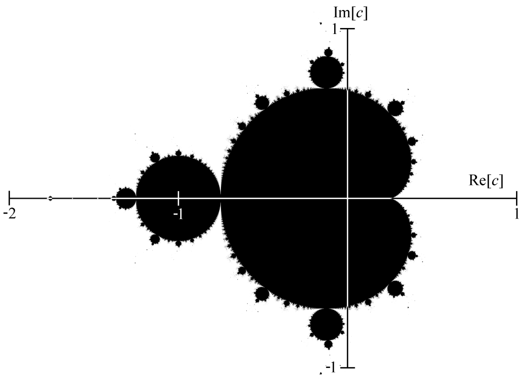
2. Let X be a topological space. A family $(V_i)_{i\in I}$ of subsets of X is said to have the **finite intersection property** if for all finite subsets J of I, the intersection $\bigcap_{j\in J} V_j$ is nonempty. Prove that X is compact if and only if it has the following property: for every family $(V_i)_{i\in I}$ of closed subsets with the finite intersection property, $\bigcap_{i\in I} V_i$ is nonempty.

3. (i) Prove that a compact Hausdorff space is regular.
   (ii) Deduce that a compact Hausdorff space is normal.

4. For $f\colon \mathbb{C}\to\mathbb{C}$ and $n\geq 0$, write f^n for the n-fold composite $f\circ\cdots\circ f$. For each $c\in\mathbb{C}$, define $f_c\colon\mathbb{C}\to\mathbb{C}$ by $f_c(z) = z^2 + c \quad (z\in\mathbb{C})$. The **Mandelbrot set** is

$$M = \big\{c\in\mathbb{C} : \text{the sequence } (f_c^n(0))_{n=0}^\infty \text{ is bounded}\big\} \subseteq \mathbb{C}.$$

   It looks like this:



   (i) Prove that if $|c| > 2$ then $c\notin M$. *(Hint: write $|c| = 2+\varepsilon$ and prove that $|f_c^n(0)| \geq 2 + n\varepsilon$ for all $n\geq 1$.)*
   (ii) Deduce that $c\in M$ if and only if $|f_c^n(0)|\leq 2$ for all $n\geq 0$.
   (iii) Deduce that the Mandelbrot set is compact.

Challenge questions

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5. Write $\{0,1\}$ for the two-point discrete topological space, and $\{0,1\}^{\mathbb{N}}$ for the product

$$\{0,1\} \times \{0,1\} \times \cdots$$

of countably infinitely many copies of $\{0,1\}$, with the product topology that you discovered in Sheet 3, q.4. Prove that $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the subspace

$$C = \left\{ x \in [0,1] : x = \sum_{n=1}^{\infty} a_n 3^{-n} \text{ for some } a_1, a_2, \ldots \in \{0,2\} \right\}$$

of $\mathbb{R}$. *(Both $\{0,1\}^{\mathbb{N}}$ and C are called the* **Cantor set**. *This question becomes easier if you allow yourself to assume a hard theorem: that for any family $(X_i)_{i \in I}$ of compact spaces, whether I is finite or not, the product space $\prod_{i \in I} X_i$ is again compact. Note that although each of the components $\{0,1\}$ is discrete, the product $\{0,1\}^{\mathbb{N}}$ is not.)*

6. Prove that every nonempty compact metrizable space is a quotient of the Cantor set. *(Hard to believe—and also hard to prove!)*

7. Do a web search for 'space-filling curve', and you'll discover that there exist sequences $(f_n)_{n=1}^{\infty}$ of continuous maps $[0,1] \to [0,1]^2$ with the properties that (i) (f_n) converges uniformly, and (ii) $\bigcup_{n=1}^{\infty} f_n[0,1]$ is dense in $[0,1]^2$. Deduce that there exists a continuous surjection $[0,1] \to [0,1]^2$. *(Hint: compactness is involved.)* Deduce from this that there exists a continuous surjection $\mathbb{R} \to \mathbb{R}^2$.

General Topology 5

Connectedness

The deadline for handing this work in is **1pm on Monday 24 November 2014**. Details of where to hand in, how the work will be assessed, etc., can be found in the FAQ on the course Learn page.

All the questions will be assessed except where noted otherwise. Please take care over communication and presentation. Your answers should be coherent, logical arguments written in full sentences. Please report mistakes on this sheet to Tom.Leinster@ed.ac.uk.

1. Let X and Y be topological spaces and let $f\colon X \to Y$ be a function. Recall that when $A \subseteq X$, we write $f|_A\colon A \to Y$ for the restriction of f to A.
   (i) Let $(U_i)_{i\in I}$ be an open cover of X. Prove that f is continuous if and only if $f|_{U_i}$ is continuous for each $i \in I$.
   (ii) Let $(V_i)_{i\in I}$ be a finite closed cover of X. Prove that f is continuous if and only if $f|_{V_i}$ is continuous for each $i \in I$.
   (iii) Give a simple example showing how the 'if' part of (ii) might be used in practice.
2. Which spaces with the cofinite topology are connected? Which are totally disconnected?
3. Let X be a nonempty topological space and let $(A_i)_{i\in I}$ be a cover of X by connected subspaces A_i. Suppose that for all $i, j \in I$, there exist $n \geq 0$ and $i_0, \ldots, i_n \in I$ such that $i_0 = i$, $i_n = j$, and
$$A_{i_0} \cap A_{i_1} \neq \emptyset, \quad A_{i_1} \cap A_{i_2} \neq \emptyset, \quad \ldots, \quad A_{i_{n-1}} \cap A_{i_n} \neq \emptyset.$$
   Prove that X is connected.
   (This is a stronger version of Lemma C1.11. Hint: draw a picture.)
4. Prove that the product of two path-connected spaces is path-connected.
5. Let $K(X)$ denote the set of connected-components of a space X. Given topological spaces X and Y, construct a bijection between $K(X \times Y)$ and $K(X) \times K(Y)$.
6. Prove that the path-components of a space X are the maximal path-connected subspaces of X. In other words, prove that each path-component C of a space X is path-connected, and that no other path-connected subspace of X contains C.

Challenge questions

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7. Let X be a Hausdorff space, and let $C_0 \supseteq C_1 \supseteq \cdots$ be an infinite sequence of compact connected subspaces of X. Show that $\bigcap_{n=0}^{\infty} C_n$ is connected.
8. Let X be a topological space and $x, y \in X$. Consider the following two conditions:
   (i) there exist disjoint open subsets U and V of X with $x \in U$, $y \in V$ and $U \cup V = X$;
   (ii) x and y are in different connected-components of X.

   Verify that (i) implies (ii). Then *(the hard part)* find an example to show that (ii) does not imply (i).

   Even if you don't do this question, there's a valuable lesson here: (i) is a stronger condition than (ii). It's tempting to believe they're equivalent, but they're not!