New Perspectives on Euler Characteristic

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1. What is Euler characteristic?

2. The Euler characteristic of a category

Credits: Schanuel, Rota, Baez–Dolan, ...
Euler characteristic is cardinality

The Euler characteristic of an object is the most basic dimensionless quantity associated with it.
Some cardinality-like invariants

Cardinality
Euler characteristic
Measure
Probability
Truth value?
Dimension?

Write them all as $|\ |$.
They all satisfy the inclusion–exclusion formulas:

$$|A \cup B| = |A| + |B| - |A \cap B|,$$
$$|\emptyset| = 0.$$
Polyconvex sets

Fix \( n \geq 0 \). A subset of \( \mathbb{R}^n \) is polyconvex if it is a finite union of compact convex subsets. A ‘measure’ is a function

\[
| \cdot | : \{ \text{polyconvex subsets of } \mathbb{R}^n \} \to \mathbb{R}
\]

satisfying

- inclusion–exclusion: \( |A \cup B| = |A| + |B| - |A \cap B|, \quad |\emptyset| = 0 \)
- invariance under rigid motions
- continuity.

**Theorem (Hadwiger)**

The only measure that is dimensionless — i.e.

\[
|\lambda A| = |A| \text{ for all } \lambda > 0 \text{ and polyconvex } A
\]

— is Euler characteristic (and its scalar multiples).
Some easy spaces

Using only inclusion–exclusion and $|\bullet| = 1$, can calculate the Euler characteristics of many spaces:

**Interval:** $\left| \begin{array}{c} 0 \bullet 1 \end{array} \right| = \left| \begin{array}{c} 0 \bullet 1/2 \end{array} \right| + \left| \begin{array}{c} 1/2 \bullet 1 \end{array} \right| - \left| \begin{array}{c} 1/2 \bullet \end{array} \right| = 2\left| \begin{array}{c} 0 \bullet 1 \end{array} \right| - 1$, giving $\left| \begin{array}{c} 0 \bullet 1 \end{array} \right| = 1$

**Two points:** $\left| \begin{array}{c} \bullet \bullet \end{array} \right| = |\bullet| + |\bullet| - |\emptyset| = 1 + 1 - 0 = 2$

**Circle:** $\left| \begin{array}{c} \bigcirc \bigcirc \end{array} \right| = \left| \begin{array}{c} \bigcirc \bullet \end{array} \right| + \left| \begin{array}{c} \bullet \bigcirc \end{array} \right| - \left| \begin{array}{c} \bullet \bullet \end{array} \right| = 1 + 1 - 2 = 0$

**$[0, 1]^2$:** $\left| \begin{array}{c} [\bullet \bullet] \end{array} \right| = \left| \begin{array}{c} [\bullet \bullet] \end{array} \right| + \left| \begin{array}{c} [\bullet \bullet] \end{array} \right| - \left| \begin{array}{c} \bullet \bullet \end{array} \right| = 1 + 1 - 1 = 1$

etc etc.
Some harder spaces

Every complex rational function $f$ has a Julia set $J(f) \subseteq \mathbb{C} \cup \{\infty\}$, its ‘zone of instability’.

Example

$$J\left(z^3 + \frac{12}{25}z + \frac{116i}{125}\right) =$$

This (probably) has Euler characteristic $1/2$, by similar calculations to those above.

Conjecture

*If* $f$ *is a rational function then* $|J(f)|$ *is a well-defined rational number, $\geq 0$.*
1. What is Euler characteristic?

2. The Euler characteristic of a category
The Euler characteristic of a category

Let $\mathcal{C}$ be a finite category with objects $c_1, \ldots, c_n$. Write $Z$ for the $n \times n$ matrix $(|\text{Hom}(c_i, c_j)|)_{i,j}$.

A weighting on $\mathcal{C}$ is an $n$-tuple $k^\bullet = (k^1, \ldots, k^n) \in \mathbb{Q}^n$ such that

$$Z \begin{pmatrix} k^1 \\ \vdots \\ k^n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$ 

A coweighting on $\mathcal{C}$ is a weighting on $\mathcal{C}^{\text{op}}$.

Lemma
If $k^\bullet$ is a weighting and $k_\bullet$ a coweighting then $\sum_i k^i = \sum_i k_i$.

Definition
Suppose that $\mathcal{C}$ admits at least one weighting and at least one coweighting. Its Euler characteristic $|\mathcal{C}|$ is $\sum k^i = \sum k_i \in \mathbb{Q}$, for any weighting $k^\bullet$ and coweighting $k_\bullet$. 
The Euler characteristic of a category (continued)

If \( Z = (|\text{Hom}(c_i, c_j)|)_{i,j} \) is invertible over \( \mathbb{Q} \), we have:

\[
Z^{-1} = \begin{pmatrix}
\bullet & \cdots & \bullet \\
\vdots & \ddots & \vdots \\
\bullet & \cdots & \bullet
\end{pmatrix}
\]

\[
\begin{array}{c|cc}
\text{SUM} & k^1 & \vdots \\
\hline
k_1 & \cdots & k_n \\
\hline
\text{SUM} & |\mathcal{C}|
\end{array}
\]

Example

If \( \mathcal{C} = (c_1 \xrightarrow{1} c_2) \) then \( Z = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \), \( Z^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \), and \( |\mathcal{C}| = 1 + -2 + 0 + 1 = 0 \) (\( = |S^1| \)).

Properties of \( |\ | \)

Respects products, sums (II) and fibrations.
If \( B\mathcal{C} \) is the classifying space of \( \mathcal{C} \) then \( |\mathcal{C}| = |B\mathcal{C}| \).
The Euler characteristic of a set

A set is a category in which the only arrows are identities.

For finite sets $S$,

$$|S| = |S|$$

— the Euler characteristic of the category $S$ is the cardinality of the set $S$. 
The Euler characteristic of a poset

A poset is a category in which each hom-set has at most one element.

For a finite poset \( P \),

\[
|P| = \sum_{n \geq 0} (-1)^n |\{ \text{chains } c_0 < c_1 < \cdots < c_n \text{ in } P \}| \in \mathbb{Z}.
\]

This is closely related to Rota’s Möbius inversion.
Digression: manifolds and orbifolds

Manifolds: there is a commutative diagram

\[
\begin{array}{ccc}
\{\text{finitely-triangulated manifolds}\} & \xrightarrow{\text{poset of simplices}} & \{\text{posets}\} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\mathbb{Z} & & \\
\end{array}
\]

To extend to orbifolds: replace posets by categories and \( \mathbb{Z} \) by \( \mathbb{Q} \):

\[
\begin{array}{ccc}
\{\text{finitely-triangulated orbifolds}\} & \xrightarrow{\text{category of simplices}} & \{\text{categories}\} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\mathbb{Q} & & \\
\end{array}
\]

(joint result with Ieke Moerdijk).
The Euler characteristic of a monoid

A monoid (＝ semigroup with identity) is a category in which there is only one object.

For finite monoids $M$,

$$|M| = 1/\text{order}(M).$$

‘Explanation’ for groups $M = G$

If $G$ acts freely on a set $S$ then $|S/G| = |S|/\text{order}(G)$.

But $G$ acts freely on the contractible space $EG$, suggesting $|BG| = 1/\text{order}(G)$. 
A groupoid is a category in which every arrow is an isomorphism.

Let $G$ be a finite groupoid. Choose one object $a_i$ from each connected-component. Then

$$|G| = \sum_i 1/\text{order} \left( \text{Aut} \left( a_i \right) \right).$$

Example

If $G = \{\text{finite sets } + \text{ bijections}\}$ then

$$|G| = \sum_{n \geq 0} 1/\text{order} \left( S_n \right) = e = 2.718\ldots.$$
and we know what the cardinality / Euler characteristic of a category is.

References and details:

Tom Leinster, ‘The Euler characteristic of a category’ (available on web).

These slides: on my web page.