# ANOTHER LOOK AT EULER CHARACTERISTIC

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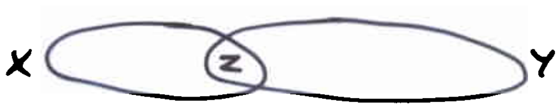
1. Euler characteristic of spaces
2. Möbius inversion
3. Euler characteristic of categories

Credits: Rota, Schanuel, Baez, ...

# 1 EULER CHARACTERISTIC OF SPACES

## What properties does $\chi$ have?

- $\chi(\emptyset) = 0$
- $\chi(\bullet) = 1$
- $\chi(X \cup_Z Y) = \chi(X) + \chi(Y) - \chi(Z)$




Also:

- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$
- $\chi(E) = \chi(B)\chi(F)$ if $E$ is fibred over $B$ with fibre $F$ (and $B$ is connected)

$\vdots$

## What do these properties imply?

We'll see what we can deduce from just these:

$$\chi(\emptyset) = 0, \qquad \chi(\bullet) = 1, \qquad \chi(X \cup_Z Y) = \chi(X) + \chi(Y) - \chi(Z).$$

- $[0,1]$: $\quad \bullet\!\!-\!\!\bullet \;\cong\; \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$

  $$\Rightarrow \chi(\bullet\!\!-\!\!\bullet) = 2\chi(\bullet\!\!-\!\!\bullet) - \chi(\bullet)$$
  $$\Rightarrow \chi(\bullet\!\!-\!\!\bullet) = 1$$

- 2-point space: $\quad \chi(\bullet\ \ \bullet) = \chi(\bullet) + \chi(\bullet) - \chi(\emptyset)$

  $$= 2$$

- $S^1$: $\quad \chi(\bigcirc) = \chi(\frown) + \chi(\smile) - \chi(\bullet\ \ \bullet)$

  $$= 0$$

- $[0,1]^2$:

  $$\Rightarrow \chi(\square) = 2\chi(\square) - \chi(|)$$
  $$\Rightarrow \chi(\square) = 1$$

- Cylinder: $\quad \chi(\ ) = \chi(\ ) + \chi(\ ) - \chi(|\,|)$

  $$= 0$$

- Torus: $\quad \chi(\ ) = 2\chi(\ ) - \chi(\bigcirc\bigcirc)$

  $$= 0$$

A lot!

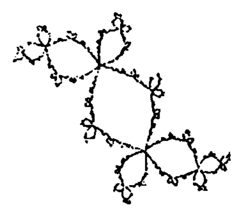
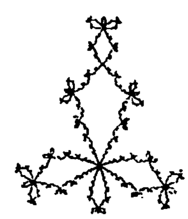
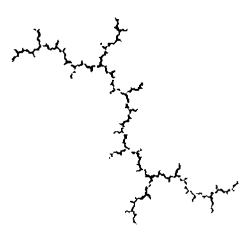
# More surprising implications

- Sierpiński's gasket, $S =$  :

$$\Rightarrow \chi(S) = 3\chi(S) - 3$$

$$\Rightarrow \chi(S) = 3/2$$

- Julia sets: every rational function $f(z) = \frac{a_0 + \dots + a_n z^n}{b_0 + \dots + b_m z^m}$ over $\mathbb{C}$ has a Julia set $J(f) \subseteq \mathbb{C} \cup \{\infty\}$.

| Rational function $f$ | $\chi(J(f))$ |
|---|---|
| $f(z) = z^2 + (-0.12 + 0.74i)$ | $\chi\Big($  $\Big) = 0$ |
| $f(z) = z^3 + \frac{12}{25} z + \frac{116}{125} i$ | $\chi\Big($  $\Big) = \frac{1}{2}$ |
| $f(z) = z^2 + i$ | $\chi\Big($  $\Big) = 1$ |

Conjecture: $\chi(J(f))$ is well-defined and $\geq 0$, for all $f$.

# How should an "Euler characteristic" behave?

**Wish:** extend $\chi$ to more general spaces.

**Idea:** $\chi$ should behave like cardinality (or measure).

Some aspects of the behaviour of cardinality of finite sets:

- If $\begin{tikzcd}[row sep=small] & X \\ Z \arrow[ur, "u"] \arrow[dr, "v"'] & \\ & Y \end{tikzcd}$ are injections then $|X \cup_Z Y| = |X| + |Y| - |Z|$

- If $X \overset{u}{\underset{v}{\rightrightarrows}} Y$ are injections with disjoint images then

$$\left| \frac{Y}{u(x) \sim v(x) \quad \forall x} \right| = |Y| - |X|$$



- If $X$ is acted on freely by a group $G$ then $|X/G| = |X| / |G|$.

"And so on."

# 2 MÖBIUS INVERSION

## Definitions

Let $\mathbb{A}$ be a finite category. Let

$$I(\mathbb{A}) = \{\text{functions } \mathrm{ob}\,\mathbb{A} \times \mathrm{ob}\,\mathbb{A} \longrightarrow \mathbb{Q}\}$$

with multiplication given by

$$(\theta\varphi)(a,c) = \sum_b \theta(a,b)\,\varphi(b,c)$$

$(\theta, \varphi \in I(\mathbb{A}),\ a, c \in \mathbb{A})$; unit is Kronecker $\delta$.

The zeta function of $\mathbb{A}$, $\zeta \in I(\mathbb{A})$, is defined by

$$\zeta(a,b) = |\mathrm{Hom}_{\mathbb{A}}(a,b)|.$$

$\mathbb{A}$ has Möbius inversion if $\zeta^{-1}$ exists; then $\mu = \zeta^{-1}$ is the Möbius function of $\mathbb{A}$.

**Examples**

- If $\mathbb{A} = \left( \begin{tikzcd}[row sep=small] & a_1 \\ a_3 \arrow[ur] \arrow[dr] & \\ & a_2 \end{tikzcd} \right)$ then $\zeta$ has matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$,

  so $\mu$ has matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$

- Let $\mathbb{D}_N$ be the cat. with objects $0, \ldots, N$, whose maps $a \to b$ are order-preserving injections $\{1, \ldots, a\} \rightarrowtail \{1, \ldots, b\}$. Then $\zeta(a,b) = \binom{b}{a}$ and $\mu(a,b) = (-1)^{b-a}\binom{b}{a}$.

- If $G$ is a group, viewed as a cat. with single object $\bigstar$, then $\zeta(\bigstar, \bigstar) = |G|$ and $\mu(\bigstar, \bigstar) = 1/|G|$.

# Digression: why the names?

| Dirichlet series | | Classical Möbius | | Möbius-Rota |
|---|---|---|---|---|
| $\{$formal D. series $\sum_{n\geq 1} \frac{\alpha_n}{n^s}$, $\alpha_n \in \mathbb{Q}\}$ | $\cong$ | $\{$sequences $(\alpha_n)_{n\geq 1}$, $\alpha_n \in \mathbb{Q}\}$ | $\longrightarrow$ | $I(\mathbb{A})$, where $\mathbb{A}$ is $\mathbb{Z}^+$ ordered by $\mid$ |
| $\sum_{n\geq 1} \frac{\alpha_n}{n^s}$ | $\longleftrightarrow$ | $(\alpha_n)_{n\geq 1}$ | $\longmapsto$ | $\bar{\alpha}$, where $\bar{\alpha}(a,b) = \begin{cases} \alpha(b/a) & \text{if } a \mid b \\ 0 & \text{if not} \end{cases}$ |
| Mult: ordinary $\cdot$ | $\longleftrightarrow$ | Mult: $(\alpha * \beta)_n = \sum_{pq=n} \alpha_p \beta_q$ | $\longmapsto$ | Mult: "ours" |
| Unit: $1$ | $\longleftrightarrow$ | Unit: $\delta$, where $\delta_n = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$ | $\longmapsto$ | Unit: Kronecker $\delta$ |
| Riemann $\zeta$ | $\longleftrightarrow$ | $\zeta$, where $\zeta_n = 1 \ \forall n$ | $\longmapsto$ | $\zeta$ of $\mathbb{A}$ |
| $\sum \frac{\mu(n)}{n^s}$ | $\longleftrightarrow$ | Classical Möbius $\mu$ | $\longmapsto$ | $\mu$ of $\mathbb{A}$ |
| $\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s}$ | $\longleftrightarrow$ | $\mu * \zeta = \delta$ | $\longmapsto$ | $\mu\zeta = \delta$ |

# The cardinality of a gluing

Glue together some finite sets. How many elements does the result have?

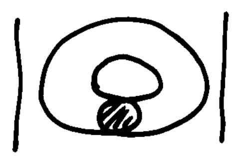
**Propn:** Let $\mathbb{A}$ be a finite category with Möbius inversion. Let $X : \mathbb{A} \longrightarrow (\text{finite sets})$ be a "good" functor. Then

$$|\operatorname{colimit}(X)| = \sum_a k_a |Xa|$$

where $k_a = \sum_b \mu(a,b)$.

**Examples:** • Let $\mathbb{A} = \left( \begin{tikzcd}[row sep=tiny] & a_1 \\ a_3 \arrow[ur] \arrow[dr] & \\ & a_2 \end{tikzcd} \right)$; then $k_{a_1} = k_{a_2} = 1$, $k_{a_3} = -1$.

A good functor $X : \mathbb{A} \longrightarrow \text{FinSet}$ is a diagram $\begin{tikzcd}[row sep=tiny] & X \\ Z \arrow[ur, "u"] \arrow[dr, "v"'] & \\ & Y \end{tikzcd}$ of finite sets where $u$ & $v$ are injections. The Propn says that $|X \cup_Z Y| = |X| + |Y| - |Z|$.

• Similarly, taking $\mathbb{A} = (a_1 \rightrightarrows a_2)$ gives 

• Let $G$ be a group. A functor $X : G \longrightarrow \text{FinSet}$ is a finite $G$-set $X$; it is good when the action is free. The Propn says that

$|X/G| = |X| / |G|$.

# 3. EULER CHARACTERISTIC OF CATEGORIES

## Definition

Let $\mathbb{A}$ be a finite category with Möbius inversion.

Its Euler characteristic is

$$\chi(\mathbb{A}) = \sum_{a,b} \mu(a,b).$$

(Can often define $\chi(\mathbb{A})$ even when $\mathbb{A}$ <u>doesn't</u> have Möbius inversion, too.)

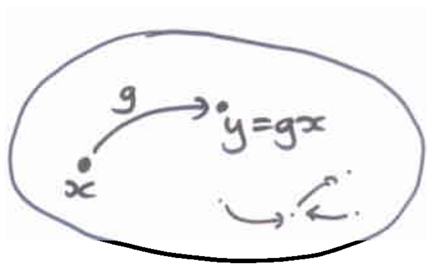
Examples:

- $\chi(\mathbb{A}) = \chi(B\mathbb{A})$ when the classifying space $B\mathbb{A}$ <u>has</u> Euler characteristic.

- If $H$ is a directed graph then
  $\chi(\text{free cat on } H) = |\text{vertices}(H)| - |\text{edges}(H)|$.

- If $G$ is a finite group then $\chi(G) = 1/|G|$.

- Let $X$ be a finite $G$-set. Let $X//G$ be the weak quotient: the cat with object-set $X$ and
  $\mathrm{Hom}_{X//G}(x,y) = \{g \in G : gx = y\}$.
  Then $\chi(X//G) = |X|/|G|$.




$X//G$

# Properties of $\chi$

Assuming that all of the following Euler characteristics exist...

- $\mathbb{A} \simeq \mathbb{B} \Rightarrow \chi(\mathbb{A}) = \chi(\mathbb{B})$
- $\chi(\mathbb{A} \times \mathbb{B}) = \chi(\mathbb{A}) \cdot \chi(\mathbb{B})$
- $\chi(\mathbb{A} \amalg \mathbb{B}) = \chi(\mathbb{A}) + \chi(\mathbb{B})$
- If $\mathbb{E}$ is fibred over $\mathbb{B}$ then $\chi(\mathbb{E})$ can be calculated from $\mathbb{B}$ and the fibres.

  (Explicitly: $\chi(\mathbb{E}) = \sum_b k_b \chi(\mathbb{F}_b)$, where $\mathbb{F}_b$ is fibre over b.)

- If there is an adjunction $\mathbb{A} \rightleftarrows \mathbb{B}$ then $\chi(\mathbb{A}) = \chi(\mathbb{B})$.

# $\chi$ of infinite categories

With care, can extend definition of $\chi$ to cover (some) infinite categories.

Example (Baez-Dolan): Let $\mathcal{E}$ be the category of finite sets and bijections. Then

$$\mathcal{E} \simeq \coprod_{n \geq 0} S_n ,$$

so

$$\begin{aligned} \chi(\mathcal{E}) &= \chi\left(\coprod_{n\geq 0} S_n\right) \\ &= \sum_{n \geq 0} \chi(S_n) \\ &= \sum_{n\geq 0} 1/n! \\ &= e . \end{aligned}$$