Sheaves do not belong to algebraic geometry

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They are, of course, very *useful* in algebraic geometry (as is the equals sign). Also, human beings discovered sheaves while developing algebraic geometry, which is why many of them still make the association.

However, we'll see that sheaves are an inevitable consequence of ideas that have nothing remotely to do with algebraic geometry. More exactly, we'll see how sheaves (and various related notions) arise automatically from two completely general categorical constructions together with one almost imperceptibly small topological observation. So sheaves not only don't belong to algebraic geometry: they don't belong to any special area of mathematics.

We will consider sheaves of sets on a topological space. To get sheaves of groups etc., just take internal groups etc. in the category of sheaves of sets.

First categorical construction

Given a small category $\mathcal{A}$, a category $\mathcal{E}$ with small colimits, and a functor $J : \mathcal{A} \longrightarrow \mathcal{E}$, there is an induced adjunction

$$\begin{tikzcd}[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \arrow[r, shift left=1.5ex, "-\otimes J"] \arrow[r, phantom, "\perp"] & \mathcal{E}. \arrow[l, shift left=1.5ex, "\mathrm{Hom}(J,-)"] \end{tikzcd}$$

Here $\perp$ means that $-\otimes J$ is left adjoint to $\mathrm{Hom}(-, J)$, and

$$(\mathrm{Hom}(J, E))(A) = \mathrm{Hom}(J(A), E)$$

$(E \in \mathcal{E},\ A \in \mathcal{A})$. The left adjoint $-\otimes J$ can be described as a certain colimit.

Example: the standard simplex functor $J : \Delta \longrightarrow \mathbf{Top}$ induces the adjunction (geometric realization, singular simplicial set) between simplicial sets and topological spaces.

Second categorical construction

Any adjunction restricts canonically to an equivalence between full subcategories.

Precisely, let $\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$ be an adjunction with unit $\eta : 1 \longrightarrow GF$ and counit $\varepsilon : FG \longrightarrow 1$. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ consisting of those objects C for which $\eta_C : C \longrightarrow GF(C)$ is an isomorphism, and dually $\mathcal{D}'$. Then the adjunction $(F, G, \eta, \varepsilon)$ restricts to an equivalence between $\mathcal{C}'$ and $\mathcal{D}'$.

Almost imperceptibly small topological input

Any open subset of a topological space can itself be regarded as a space, and any inclusion of open subsets induces a map between the corresponding spaces.

Precisely, let S be a topological space. Write $\mathbf{Open}(S)$ for the poset of open subsets of S, regarded as a category (in which each hom-set has at most one element). Write $\mathbf{Top}/S$ for the category of spaces over S (whose objects are continuous maps into S and whose morphisms are commutative triangles). Then there is a canonical functor $J : \mathbf{Open}(S) \longrightarrow \mathbf{Top}/S$, sending an open set U to the inclusion $U \hookrightarrow S$.

Finale

Fix a topological space S. The category $\mathbf{Top}/S$ has small colimits, since $\mathbf{Top}$ does.

Applying the first categorical construction to the functor J above gives an adjunction

$$\text{(presheaves on } S) = [\mathbf{Open}(S)^{\mathrm{op}}, \mathbf{Set}] \underset{\perp}{\rightleftarrows} \mathbf{Top}/S = \text{(spaces over } S).$$

The two functors here are the ones you'd guess.

Applying the second construction now gives an equivalence, namely

$$\text{(sheaves on } S) = \mathbf{Sh}(S) \rightleftarrows \mathbf{Et}(S) = \text{(étale spaces over } S).$$

This can be interpreted as a theorem or a definition, according to taste.

Going right and then left in the adjunction gives the associated sheaf (sheafification) of a presheaf. Going left and then right gives the 'étalification' of a space over S.

Challenge

Do something analogous for (a) sheaves on a site, and (b) stacks.