Magnitude homology

Tom Leinster

Edinburgh

A theme of this conference so far

When introducing a piece of category theory during a talk:

1. apologize;
2. blame John Baez.

A theme of this conference so far

When introducing a piece of category theory during a talk:

1. apologize;
2. bla

A theme of this conference so far

When introducing a piece of category theory during a talk:

Plan

1. The idea of magnitude

2. The magnitude of a metric space

3. The idea of magnitude homology

4. The magnitude homology of a metric space

1. The idea of magnitude

Size

For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies

$$\begin{aligned} |X \cup Y| &= |X| + |Y| - |X \cap Y| \\ |X \times Y| &= |X| \times |Y|. \end{aligned}$$

- Subsets of $\mathbb{R}^n$ have volume. It satisfies

$$\begin{aligned} \mathrm{vol}(X \cup Y) &= \mathrm{vol}(X) + \mathrm{vol}(Y) - \mathrm{vol}(X \cap Y) \\ \mathrm{vol}(X \times Y) &= \mathrm{vol}(X) \times \mathrm{vol}(Y). \end{aligned}$$

- Topological spaces have Euler characteristic. It satisfies

$$\begin{aligned} \chi(X \cup Y) &= \chi(X) + \chi(Y) - \chi(X \cap Y) \quad \text{(under hypotheses)} \\ \chi(X \times Y) &= \chi(X) \times \chi(Y). \end{aligned}$$

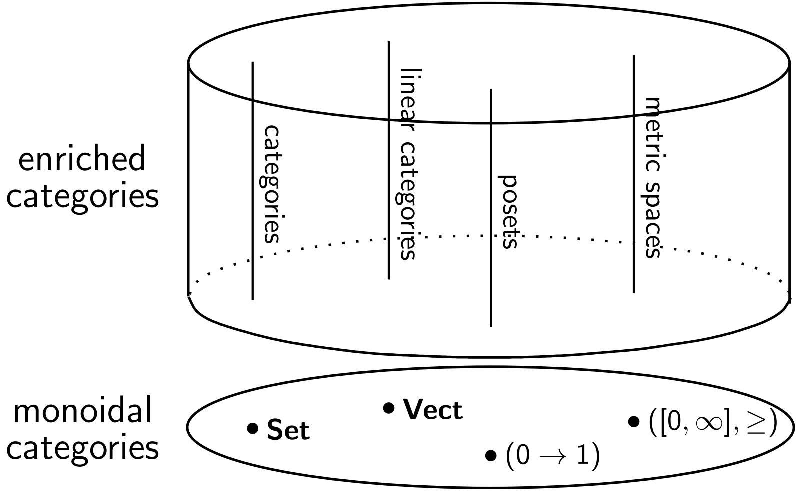
Challenge Find a general definition of 'size', including these and other examples.

One answer The magnitude of an enriched category.

Enriched categories

A monoidal category is a category $\mathscr{V}$ equipped with a product operation.

A category $\mathbf{X}$ enriched in $\mathscr{V}$ is like an ordinary category, but each $\mathrm{Hom}_{\mathbf{X}}(X, Y)$ is now an object of $\mathscr{V}$ (instead of a set).



The magnitude of an enriched category

There is a general definition of the magnitude $|\mathbf{X}|$ of an enriched category. (Definition and details omitted.)

Examples This gives definitions of:

- the magnitude of a poset
- the magnitude of an ordinary category
- the magnitude of a linear category
- the magnitude of a metric space.

The magnitude of a poset

The magnitude of a poset is better known as its Euler characteristic (1960s).

Example Let M be a triangulated manifold.

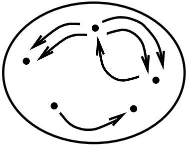
Write P for the poset of simplices in the triangulation, ordered by inclusion.

Then

$$|P| = \chi(M).$$

The magnitude of an ordinary category

Let **X** be a finite category.



'Recall': **X** gives rise to a topological space $B\mathbf{X}$ (its classifying space), built as follows:

- for each object of **X**, put a point • into $B\mathbf{X}$;
- for each map $x \to y$ in **X**, put an interval •——• into $B\mathbf{X}$;
- for each commutative triangle in **X**, put a 2-simplex ▲ into $B\mathbf{X}$;
- ...

Theorem Let **X** be a finite category. Then

$$|\mathbf{X}| = \chi(B\mathbf{X}),$$

under hypotheses ensuring that $\chi(B\mathbf{X})$ is well-defined.

The magnitude of a linear category

. . . is related to the Euler form in commutative algebra.

The magnitude of a metric space

. . . is something new!

2. The magnitude of a metric space

— done explicitly —

The magnitude of a finite metric space

Let X be a finite metric space.

Write Z_X for the $X \times X$ matrix with entries

$$Z_X(x,y) = e^{-d(x,y)}$$

$(x, y \in X)$. (Why $e^{-\text{distance}}$? Because $e^{-(u+v)} = e^{-u}e^{-v}$.)

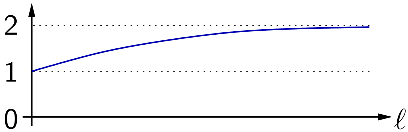
If Z_X is invertible (which it is if $X \subseteq \mathbb{R}^n$), the magnitude of X is

$$|X| = \sum_{x,y \in X} Z_X^{-1}(x,y) \ \in \mathbb{R}$$

—the sum of all the entries of the inverse matrix of Z_X.

First examples

- $|\emptyset| = 0$.
- $|\bullet| = 1$.
- $\left|\bullet\!\overset{\leftarrow\ \ell\ \rightarrow}{}\!\bullet\right| =$ sum of entries of $\begin{pmatrix} e^{-0} & e^{-\ell} \\ e^{-\ell} & e^{-0} \end{pmatrix}^{-1} = \dfrac{2}{1 + e^{-\ell}}$

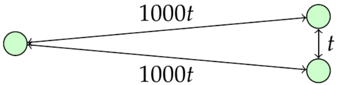


- If $d(x, y) = \infty$ for all $x \neq y$ then $|X| =$ cardinality(X).

Slogan: Magnitude is the 'effective number of points'.

Example: a 3-point space (Simon Willerton)

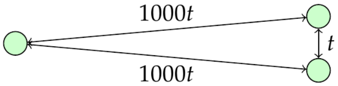
Take the 3-point space

$X =$ 

- When t is small, X looks like a 1-point space.

Example: a 3-point space (Simon Willerton)

Take the 3-point space

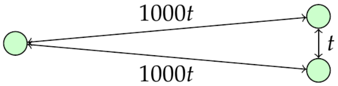
$X =$ 

- When t is small, X looks like a 1-point space.
- When t is moderate, X looks like a 2-point space.

Example: a 3-point space (Simon Willerton)

Take the 3-point space

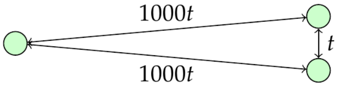
$X =$



- When t is small, X looks like a 1-point space.
- When t is moderate, X looks like a 2-point space.
- When t is large, X looks like a 3-point space.

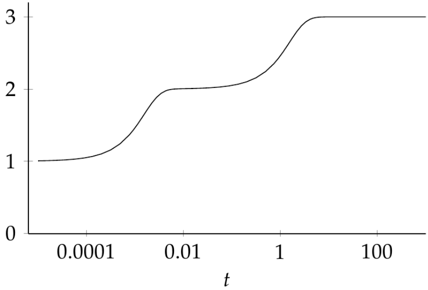
Example: a 3-point space (Simon Willerton)

Take the 3-point space

$X =$ 

- When t is small, X looks like a 1-point space.
- When t is moderate, X looks like a 2-point space.
- When t is large, X looks like a 3-point space.

Indeed, the magnitude of X as a function of t is:



Magnitude functions

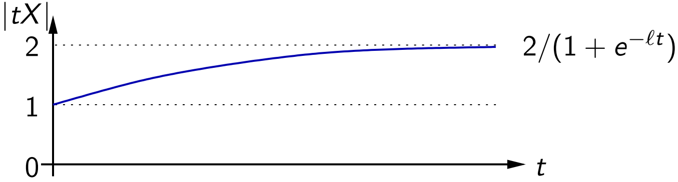
Magnitude assigns to each metric space not just a *number*, but a *function*.

For $t > 0$, write tX for X scaled up by a factor of t.

The magnitude function of a metric space X is the partially-defined function

$$\begin{array}{ccc} (0,\infty) & \to & \mathbb{R} \\ t & \mapsto & |tX|\,. \end{array}$$

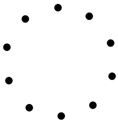
E.g.: the magnitude function of $X = (\bullet \overset{\leftarrow\ \ell\ \rightarrow}{} \bullet)$ is



A magnitude function has only finitely many singularities (none if $X \subseteq \mathbb{R}^n$).

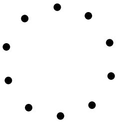
It is increasing for $t \gg 0$, and $\lim_{t\to\infty} |tX| = \text{cardinality}(X)$.

Dimension at different scales

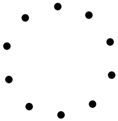
Let $X =$  , with subspace metric from $\mathbb{R}^2$.

- When t is small, tX looks 0-dimensional.

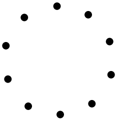
Dimension at different scales

Let $X =$  , with subspace metric from $\mathbb{R}^2$.

- When t is small, tX looks 0-dimensional.
- When t is moderate, tX looks nearly 1-dimensional.

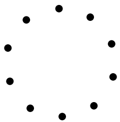


Dimension at different scales

Let $X =$  , with subspace metric from $\mathbb{R}^2$.

- When t is small, tX looks 0-dimensional.
- When t is moderate, tX looks nearly 1-dimensional.
- When t is large, tX looks 0-dimensional again.

Dimension at different scales

Let $X =$  , with subspace metric from $\mathbb{R}^2$.

- When t is small, tX looks 0-dimensional.
- When t is moderate, tX looks nearly 1-dimensional.
- When t is large, tX looks 0-dimensional again.

The magnitude function sees all this!

Here's how. . .

Dimension at different scales (Willerton)

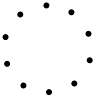
For a function $f\colon (0,\infty) \to \mathbb{R}$, the instantaneous growth of f at $t \in (0,\infty)$ is

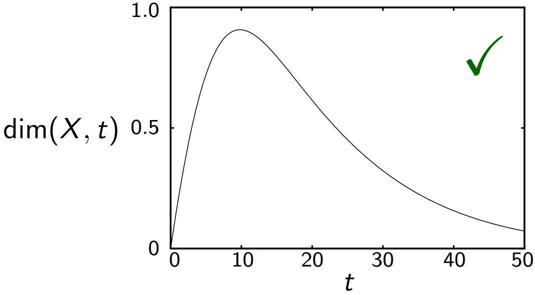
$$\text{growth}(f,t) = \frac{d(\log f(t))}{d(\log t)} = \text{slope of the log-log graph of } f \text{ at } t.$$

E.g.: If $f(t) = Ct^n$ then $\text{growth}(f,t) = n$ for all t.

For a space X, the magnitude dimension of X at scale t is

$$\dim(X,t) = \text{growth}(|tX|, t).$$

E.g.: Let $X =$  .



The magnitude of a compact metric space

A metric space M is positive definite if for every finite $Y \subseteq M$, the matrix Z_Y is positive definite.

E.g.: $\mathbb{R}^n$ with Euclidean or taxicab metric; sphere with geodesic metric; hyperbolic space; any ultrametric space.

Theorem (Mark Meckes)
All sensible ways of extending the definition of magnitude from finite metric spaces to compact positive definite spaces are equivalent.

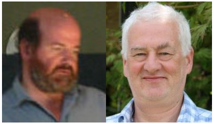
For a compact positive definite space X,

$$|X| = \sup\{|Y| \; : \; \text{finite } Y \subseteq X\}.$$

Magnitude encodes geometric information

Theorem (Juan-Antonio Barceló & Tony Carbery)

For compact $X \subseteq \mathbb{R}^n$,



$$\mathrm{vol}_n(X) = C_n \lim_{t \to \infty} \frac{|tX|}{t^n}$$

where C_n is a known constant.

Theorem (Heiko Gimperlein & Magnus Goffeng)

Assume n is odd. For 'nice' compact $X \subseteq \mathbb{R}^n$
(meaning that ∂X is smooth and $\mathrm{Cl}(\mathrm{Int}(X)) = X$),

$$|tX| = c_n \,\mathrm{vol}_n(X) t^n + c_{n-1} \,\mathrm{vol}_{n-1}(\partial X) t^{n-1} + O(t^{n-2})$$

as $t \to \infty$, where c_n and c_{n-1} are known constants.

The magnitude function knows the volume and the surface area.

Magnitude encodes geometric information

Magnitude satisfies an asymptotic inclusion-exclusion principle:

Theorem (Gimperlein & Goffeng)

Assume n is odd. Let $X, Y \subseteq \mathbb{R}^n$ with X, Y and $X \cap Y$ nice. Then

$$|t(X \cup Y)| + |t(X \cap Y)| - |tX| - |tY| \to 0$$

as $t \to \infty$.

But not all results are asymptotic! Let B^n denote the unit ball in $\mathbb{R}^n$.

Theorem (Barceló & Carbery; Willerton)

Assume n is odd. Then $|tB^n|$ is a known rational function of t over $\mathbb{Z}$.

Examples

- $\left|tB^1\right| = |[-t, t]| = 1 + t$
- $\left|tB^3\right| = 1 + 2t + t^2 + \frac{1}{3!}t^3$
- $\left|tB^5\right| = \dfrac{24 + 72t^2 + 35t^3 + 9t^4 + t^5}{8(3 + t)} + \dfrac{t^5}{5!}$.

3. The idea of magnitude homology

Two perspectives on Euler characteristic

So far: Euler characteristic has been treated as an analogue of cardinality.

Alternatively: Given any homology theory H_* of any kind of object X, can define

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(X).$$

Note:

- $\chi(X)$ is a *number*
- $H_*(X)$ is an *algebraic structure*, and functorial in X.

We say that H_* is a categorification of χ.

So, homology categorifies Euler characteristic.

Towards magnitude homology of enriched categories

Any *ordinary* category $\mathbf{X}$ gives rise to a chain complex $C_*(\mathbf{X})$:

$$C_n(\mathbf{X}) = \coprod_{x_0,\ldots,x_n \in \mathbf{X}} \mathbb{Z} \cdot \big(\mathbf{X}(x_0, x_1) \times \cdots \times \mathbf{X}(x_{n-1}, x_n)\big)$$

where $\mathbb{Z} \cdot - \colon \mathbf{Set} \to \mathbf{Ab}$ is the free abelian group functor.

The homology $H_*(\mathbf{X})$ of $\mathbf{X}$ is the homology of $C_*(\mathbf{X})$.

Theorem $H_*(\mathbf{X}) = H_*(B\mathbf{X})$.

Generalizing this, Michael Shulman gave a definition of the homology of an *enriched* category (omitted here).

It should categorify magnitude.

In particular, it gives a homology theory of metric spaces. . .

4. The magnitude homology of a metric space

Special case of graphs: Hepworth and Willerton (2015)

General case of enriched categories: Shulman (*n*-Category Café, Aug 2016)

The shape of the definition

In this talk, a persistence module is a functor

$$A \colon ([0,\infty], \geq) \to \mathbf{Ab}.$$

That is: it's a family $(A(\ell))_{\ell \in [0,\infty]}$ of abelian groups, together with a homomorphism $\alpha_{\ell,k} \colon A(\ell) \to A(k)$ whenever $\ell \geq k$, such that $\alpha_{\ell,k} \circ \alpha_{m,\ell} = \alpha_{m,k}$ and $\alpha_{\ell,\ell} = \mathrm{id}$.

We will define the homology

$$H_*(X, A)$$

of a metric space X with coefficients in a persistence module A. Each $H_n(X, A)$ is an abelian group.

This is a special case of the general definition for enriched categories.

The definition

Let X be a metric space and let A be a persistence module.

There is a chain complex $C(X, A)$ with

$$C_n(X,A) = \coprod_{x_0,\dots,x_n \in X} A\big(d(x_0,x_1) + \cdots + d(x_{n-1},x_n)\big).$$

The differential is

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i \colon C_n(X,A) \to C_{n-1}(X,A)$$

where (e.g.) in the case $n = 2$, the maps $\partial_0, \partial_1, \partial_2$ are given as follows:

the inequality $d(x_0,x_1) + d(x_1,x_2) \geq d(x_1,x_2)$ induces
a homomorphism $\partial_0 \colon A\big(d(x_0,x_1) + d(x_1,x_2)\big) \to A\big(d(x_1,x_2)\big)$.

The definition

Let X be a metric space and let A be a persistence module.

There is a chain complex $C(X, A)$ with

$$C_n(X, A) = \coprod_{x_0, \dots, x_n \in X} A\big(d(x_0, x_1) + \cdots + d(x_{n-1}, x_n)\big).$$

The differential is

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i \colon C_n(X, A) \to C_{n-1}(X, A)$$

where (e.g.) in the case $n = 2$, the maps $\partial_0, \partial_1, \partial_2$ are given as follows:

> the inequality $d(x_0, x_1) + d(x_1, x_2) \geq d(x_0, x_2)$ induces
> a homomorphism $\partial_1 \colon A\big(d(x_0, x_1) + d(x_1, x_2)\big) \to A\big(d(x_0, x_2)\big)$.

The definition

Let X be a metric space and let A be a persistence module.

There is a chain complex $C(X, A)$ with

$$C_n(X, A) = \coprod_{x_0, \ldots, x_n \in X} A\big(d(x_0, x_1) + \cdots + d(x_{n-1}, x_n)\big).$$

The differential is

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i \colon C_n(X, A) \to C_{n-1}(X, A)$$

where (e.g.) in the case $n = 2$, the maps $\partial_0, \partial_1, \partial_2$ are given as follows:

the inequality $d(x_0, x_1) + d(x_1, x_2) \geq d(x_0, x_1)$ induces
a homomorphism $\partial_2 \colon A\big(d(x_0, x_1) + d(x_1, x_2)\big) \to A\big(d(x_0, x_1)\big)$.

The magnitude homology $H_*(X, A)$ is the homology of $C_*(X, A)$.

Magnitude homology with coefficients in a point module

For each $\ell \in [0, \infty]$, define a persistence module A_ℓ by

$$A_\ell(k) = \begin{cases} \mathbb{Z} & \text{if } k = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$C_n(X, A_\ell) = \mathbb{Z} \cdot \{(x_0, \ldots, x_n) : d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell\}.$$

Equivalently, we can replace $C_*(X, A)$ by a normalized version:

$$C_n^\sharp(X, A_\ell) = \mathbb{Z} \cdot \{(x_0, \ldots, x_n) : d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell,\ x_0 \neq \cdots \neq x_n\}.$$

The differential is $\partial = \sum_{i=1}^{n-1} (-1)^i \partial_i$, where

$$\partial_i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{if } x_i \text{ is between } x_{i-1} \text{ and } x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

'Between' means that $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1})$.

H_1 detects convexity

A metric space X is Menger convex if for all distinct $x, y \in X$, there exists $z \in X$ between x and y with $x \neq z \neq y$.

Theorem Let X be a metric space. Then

$$X \text{ is Menger convex} \iff H_1(X, A_\ell) = 0 \text{ for all } \ell > 0.$$

Corollary Let X be a closed subset of $\mathbb{R}^n$. Then

$$X \text{ is convex} \iff H_1(X, A_\ell) = 0 \text{ for all } \ell > 0.$$

And, for instance, if

$$X = (\bullet \qquad \bullet \;\; \bullet \quad \bullet \quad \bullet \qquad \bullet \quad \bullet) \subseteq \mathbb{R}$$

with all gaps of length $< \varepsilon$, then $H_1(X, A_\ell) = 0$ for all $\ell \geq \varepsilon$.

Back to Euler characteristic

Let X be a metric space. For any persistence module A, put

$$\chi(X, A) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(X, A)$$

(if defined). In particular, we have an Euler characteristic

$$\chi(X, A_\ell) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(X, A_\ell)$$

for each $\ell \in [0, \infty)$. Not just one Euler characteristic: many!

Make these Euler characteristics into the coefficients of a formal expression:

$$\chi(X) = \sum_{\ell \in [0,\infty)} \chi(X, A_\ell)\, q^\ell.$$

Claim: $\chi(X)$ is formally equal to $|tX|$, where $q = e^{-t}$. So I'm claiming:

Magnitude homology categorifies magnitude

Magnitude homology categorifies magnitude: ‘proof’

$$
\begin{aligned}
\chi(X) &= \sum_{\ell \in [0,\infty)} \sum_{n \in \mathbb{N}} (-1)^n \operatorname{rank} H_n(X, A_\ell)\, q^\ell = \sum_{\ell, n} (-1)^n \operatorname{rank} C_n^\sharp(X, A_\ell)\, q^\ell \\
&= \sum_{n,\ell} (-1)^n \Big| \big\{ (x_0, \ldots, x_n) : d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell,\, x_0 \neq \cdots \neq x_n \big\} \Big| q^\ell \\
&= \sum_n (-1)^n \sum_{x_0, \ldots, x_n :\, x_0 \neq \cdots \neq x_n} q^{d(x_0,x_1) + \cdots + d(x_{n-1},x_n)} \\
&= \sum_n (-1)^n \sum_{x_0, \ldots, x_n \in X} \big(q^{d(x_0,x_1)} - \delta_{x_0,x_1} \big) \cdots \big(q^{d(x_{n-1},x_n)} - \delta_{x_{n-1},x_n} \big) \\
&= \sum_n (-1)^n \sum_{x_0, \ldots, x_n \in X} (Z_{tX} - I)_{x_0,x_1} \cdots (Z_{tX} - I)_{x_{n-1},x_n} \\
&= \sum_n (-1)^n \operatorname{sum} \big((Z_{tX} - I)^n \big) \quad \text{where sum means the sum of all entries} \\
&= \operatorname{sum} \Big(\sum_{n \in \mathbb{N}} (I - Z_{tX})^n \Big) = \operatorname{sum} \Big(\big(I - (I - Z_{tX}) \big)^{-1} \Big) = \operatorname{sum} \big(Z_{tX}^{-1} \big) \\
&= |tX|. \quad \checkmark
\end{aligned}
$$

…formally, at least!

Open questions

1. What information does the magnitude homology $H_*(X, A)$ capture when we use other persistence modules A as our coefficients (e.g. interval modules)?
2. What is the relationship between magnitude homology and persistent homology?
3. Which theorems about magnitude of metric spaces can be categorified to give theorems about magnitude homology?

   Compare:

   - Many theorems about *topological* Euler characteristic are shadows of theorems about homology.
   - For the special case of graphs, Hepworth and Willerton already proved a Künneth theorem (categorifying the formula for $|X \times Y|$) and a Mayer–Vietoris theorem (categorifying formula for $|X \cup Y|$).

References (titles are clickable links)

General references on magnitude: • Leinster, The magnitude of metric spaces

• Leinster and Meckes, The magnitude of a metric space: from category theory to geometric measure theory

Magnitude of finite metric spaces (in direction of data): • Willerton, Spread: a measure of the size of metric spaces

• Willerton, Instantaneous dimension of finite metric spaces via magnitude and spread

Analytic aspects of magnitude: • Meckes, Positive definite metric spaces

• Meckes, Magnitude, diversity, capacities, and dimensions of metric spaces

• Barceló and Carbery, On the magnitudes of compact sets in Euclidean spaces

• Willerton, The magnitude of odd balls via Hankel determinants of reverse Bessel polynomials

• Gimperlein and Goffeng, On the magnitude function of domains in Euclidean space

Magnitude homology: • Hepworth and Willerton, Categorifying the magnitude of a graph

• Shulman, Leinster, et al., Magnitude homology