Entropy, diversity and magnitude

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Some measures of size

topological spaces
Euler char

posets
Euler char

sets
cardinality

groupoids
cardinality

ecological communities
diversity

orbifolds
Euler char

manifolds
Euler char

graphs
Euler char

groups
order

monoids
order

convex sets
volume
perimeter
etc.

probability distributions
entropy

associative algebras
Euler char

Plan

See how all these link up:

1. Size

2. Spread

1. Size

The magnitude of a matrix

Let Z be a square matrix.

A weighting on Z is a column vector w such that

$$Zw = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

A coweighting on Z is a row vector v such that

$$vZ = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}.$$

Now suppose Z admits both a weighting w and a coweighting v.
The magnitude of Z is

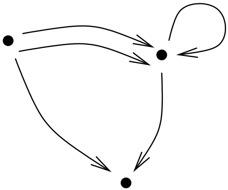
$$|Z| = \sum_i w_i = \sum_j v_j.$$

This is independent of choice of w and v.

Example: Usually Z is invertible. Then $|Z| = \sum_{i,j} (Z^{-1})_{ij}$.

The Euler characteristic of a category

A category is a directed graph with a composition law.



We'll deal with *finite* categories (those with only finitely many objects and arrows).

Let $\mathbf{A}$ be a finite category, with objects $a_1, \ldots, a_n$.

Write $Z_{\mathbf{A}}$ for the $n \times n$ matrix with

$$(Z_{\mathbf{A}})_{ij} = |\mathrm{Hom}(a_i, a_j)| = (\text{number of arrows from } a_i \text{ to } a_j).$$

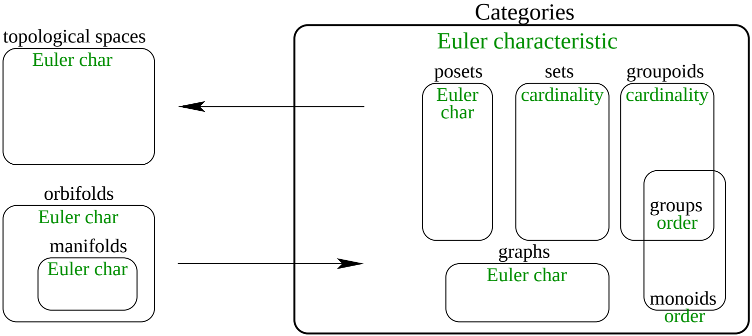
The Euler characteristic (or magnitude) of $\mathbf{A}$ is $|\mathbf{A}| = |Z_{\mathbf{A}}| \in \mathbb{Q}$.

Example

Let $\mathbf{A} = (\bullet \rightrightarrows \bullet)$. Then $Z_{\mathbf{A}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, so $(Z_{\mathbf{A}})^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, so

$$|\mathbf{A}| = 1 + (-2) + 0 + 1 = 0.$$

Euler characteristic of categories links up invariants of size



There is a series of comparison theorems (under size hypotheses). E.g.:

- Every set A can be viewed as a category $\mathbf{A}$, with only identity maps. Then $|A| = |\mathbf{A}|$.
- Every category $\mathbf{A}$ gives rise to a topo space $B\mathbf{A}$ (its classifying space). Then $\chi(B\mathbf{A}) = |\mathbf{A}|$.
- Every triangulated orbifold A gives rise to a category $\mathbf{A}$. Then $\chi(A) = |\mathbf{A}|$ (not necessarily an integer).

Casting the net wider: enriched categories

In a *category*, to any pair (a, b) of objects there is assigned a set (namely, the set of maps from a to b).

In an *enriched category*, to any pair (a, b) of objects there is assigned...
...a something.

Examples

- In a linear category, to any pair (a, b) of objects there is assigned a vector space $\mathrm{Hom}(a, b)$.
- In a metric space, to any pair (a, b) of objects (points) there is assigned a real number $d(a, b)$.

The magnitude of an enriched category

Strategy: We can define the magnitude of an enriched category by imitating the definition for ordinary categories.

Example: the magnitude of a linear category

Let $\mathbf{A}$ be a linear category with objects $a_1, \ldots, a_n$.
Write $Z_{\mathbf{A}}$ for the $n \times n$ matrix with

$$(Z_{\mathbf{A}})_{ij} = \dim(\mathrm{Hom}(a_i, a_j)).$$

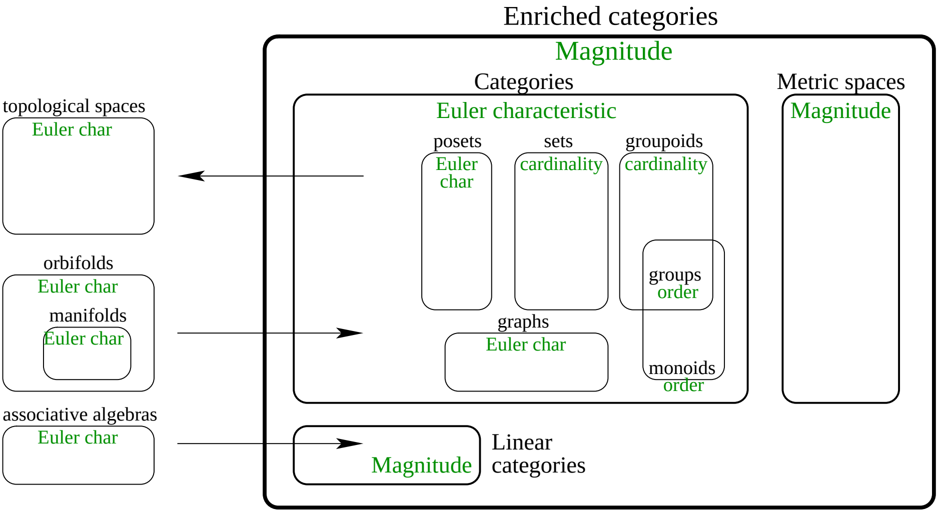
The magnitude of $\mathbf{A}$ is $|\mathbf{A}| = |Z_{\mathbf{A}}|$.

Theorem (with Catharina Stroppel)

Let A be a Koszul algebra satisfying finiteness conditions.
Write $\mathbf{A}$ for the linear category of projective indecomposable A-modules.
Then

$$|\mathbf{A}| = \sum_{i=0}^{\infty} (-1)^i \dim(\mathrm{Ext}^i_A(A_0, A_0)).$$

The magnitude of an enriched category



The magnitude of a finite metric space

Let $A = \{a_1, \ldots, a_n\}$ be a finite metric space.
Write Z_A for the $n \times n$ matrix with

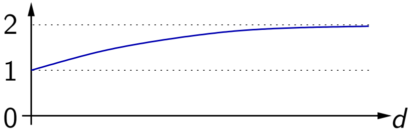
$$(Z_A)_{ij} = e^{-d(a_i, a_j)}.$$

The magnitude of A is $|A| = |Z_A| \in \mathbb{R}$.

Explicitly: $|A| = \sum_{i,j} (Z_A^{-1})_{i,j}$ (which is almost always defined).

Examples

- $|\emptyset| = 0$ and $|\bullet| = 1$.
- $|\bullet \xleftarrow{} d \xrightarrow{} \bullet| = 1 + \tanh(d/2)$:



- If $d(a_i, a_j) = \infty$ for all $i \neq j$ then $|A| = n$.

The magnitude of a compact metric space

Compact metric spaces can be approximated by finite spaces.

This lets us extend the definition of magnitude to (nice) compact spaces.

Theorem (Mark Meckes)

Let A be a compact metric space, and 'nice' (e.g. a subset of $\mathbb{R}^n$).

Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ be finite subspaces with $\overline{\bigcup A_i} = A$.

Then $\lim_{i\to\infty} |A_i|$ exists and depends only on A (not on (A_i)).

The magnitude $|A|$ of A can then be defined as $\lim_{i\to\infty} |A_i|$.

The geometric significance of magnitude

Example: $|[0, \ell]| = 1 + \ell/2$.

So, *the magnitude of a line tells you its length*.

Let A be a compact metric space.
Given $t > 0$, write tA for A scaled by a factor of t.

The magnitude function of A is the (partially defined) function

$$\begin{array}{ccc} (0, \infty) & \to & \mathbb{R}, \\ t & \mapsto & |tA|. \end{array}$$

Example: The magnitude function of $[0, \ell]$ is $t \mapsto 1 + (\ell/2) \cdot t$.

The convex magnitude conjecture (in dimension 2)

Conjecture (with Simon Willerton)

Let $A \subseteq \mathbb{R}^2$ be a compact convex set. Then

$$|A| = \chi(A) + \frac{1}{4}\mathit{perimeter}(A) + \frac{1}{2\pi}\mathit{area}(A).$$

If true, then

$$|tA| = \chi(A) + \frac{1}{4}\text{perimeter}(A) \cdot t + \frac{1}{2\pi}\text{area}(A) \cdot t^2.$$

So:

the magnitude function of a convex planar set
tells you its Euler characteristic, perimeter and area.

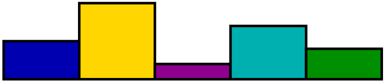
Moreover, the degree of the polynomial is the dimension of the space.

2. Spread

Introduction to diversity and entropy

Consider an ecological community of n species, in proportions $p_1, \ldots, p_n$.

(Thus, $p_i \geq 0$ and $\sum p_i = 1$.) Write $p = (p_1, \ldots, p_n)$.



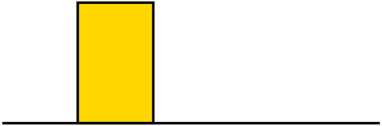
One measure of the community's diversity is

$$D(p) = \prod_{i=1}^{n} p_i^{-p_i}$$

—the exponential of Shannon entropy.

Max value is n, when $p_i = 1/n$ for all i:

Min value is 1, when $p_i = 1$ for some i:



A refined ecological model

A diversity measure should also reflect the varying *differences* between species.

Suppose we have, for each $i, j \in \{1, \ldots, n\}$, a number

$$0 \leq Z_{ij} \leq 1$$

indicating the similarity (e.g. genetic) between the ith and jth species.

Here $Z_{ij} = 0$ means total dissimilarity, and $Z_{ij} = 1$ means identical species.

This gives an $n \times n$ 'similarity matrix' Z, symmetric and with $Z_{ii} = 1$.

Example

The naive model: different species never have anything in common.

Then $Z = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I$.

A refined diversity measure

Take a community divided into n species, in proportions $p = (p_1, \ldots, p_n)$, with similarity matrix Z.

One measure of the community's diversity is

$$D^Z(p) = \prod_{i=1}^{n} (Zp)_i^{-p_i}.$$

Example

In the naive model ($Z = I$), we have

$$D^Z(p) = D(p) = \exp(\text{Shannon entropy}).$$

But in general, diversity is *not* maximized by the uniform distribution.

How to maximize diversity

Suppose we have a list of n species, with known similarity matrix Z.

- What is the maximum diversity, $\max_p D^Z(p)$?
- Which distribution p maximizes the diversity?

Theorem

Under hypotheses on the matrix Z:

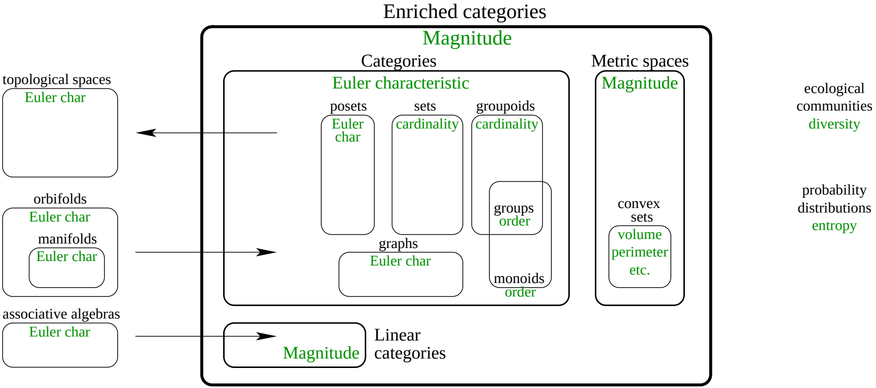
- *The maximum diversity is the magnitude $|Z|$.*
- *The maximizing distribution is the unique weighting on Z, normalized.*

(Under no hypotheses at all, something similar but more complicated holds.)

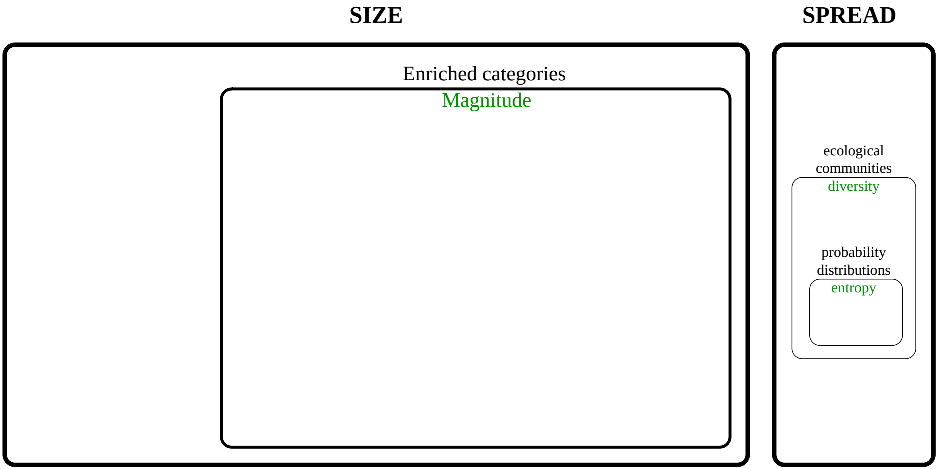
So, very roughly: *magnitude is maximum diversity*.

Summary

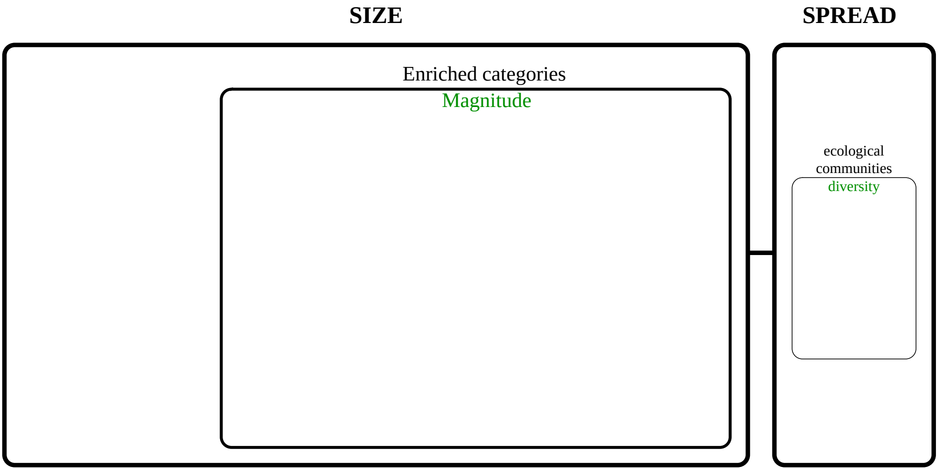
Invariants of size and spread



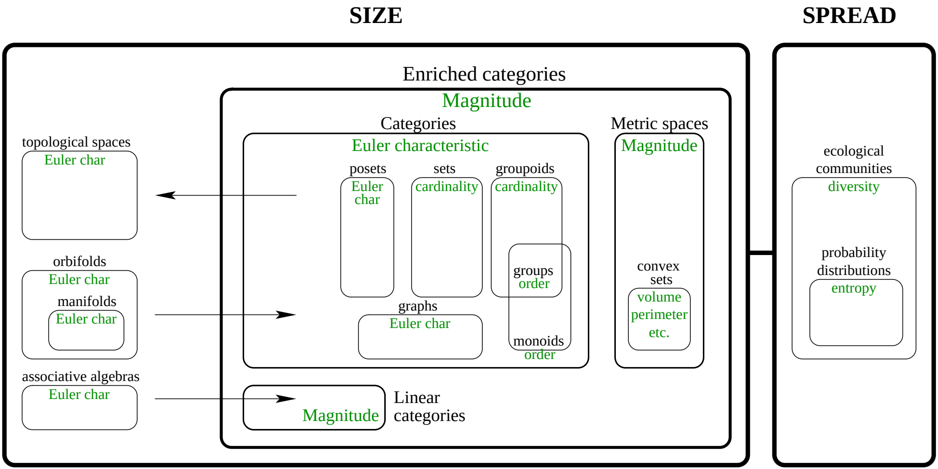
Invariants of size and spread



Invariants of size and spread



Invariants of size and spread



Thank you