Flatness of functors into sites

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Filteredness is geometric

Consider a functor F on a small category $\mathcal{C}$.

Filteredness of (the dual of) the category of elements of F is expressible in geometric logic in the language of functors on $\mathcal{C}$:

- $\bigvee_{C \in \mathcal{C}} \exists x \colon C \ (x = x)$

- $\forall x' \colon C' \ \forall x'' \colon C'' \ \bigvee_{C} \bigvee_{\substack{u' \colon C \longrightarrow C' \\ u'' \colon C \longrightarrow C''}} \exists x \colon C \ (u'(x) = x' \wedge u''(x) = x'')$

- $\forall x \colon C' \ (u(x) = v(x) \longrightarrow \ \bigvee_{C} \bigvee_{\substack{w \colon C \longrightarrow C' \\ u \circ w = v \circ w}} \exists y \colon C \ (w(y) = x))$

Thus it can be attributed to functors with values in any site $(\mathcal{K}, j)$

Examples: The *filtering* functors from a weakly lex category to an exact one (equipped with the topology of singleton epi-coverings), the *multilimit-merging* functors from a familially lex category to a lextensive one (equipped with the topology of sums), the *fm-limit-merging* functors from an fm-complete* category to a pretopos (equipped with the precanonical topology)

* it means that, for all finite diagrams D the cone functor

Cone$(D) \colon \mathcal{C}^{op} \longrightarrow$ Set is finite colimit of representables

Left exact Kan extensions

In all the above examples filteredness implies flatness, i.e the left Kan extension

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "y"] \arrow[dr, "F"'] & {[\mathcal{C}^{op}, \mathsf{Set}]} \arrow[d, "\mathsf{Lan}_y F"] \\
& \mathcal{K}
\end{tikzcd}
$$

preserves finite limits (and so do the restrictions of $\mathsf{Lan}_y F$ to the exact, lextensive, pretopos completion of $\mathcal{C}$, respectively).

This happens because the colimits used in the calculation of $\mathsf{Lan}_y F$ have the correct behaviour: They are *postulated* in the sense of A. Kock, i.e they satisfy in the internal logic of $(\mathcal{K}, j)$, for all diagrams $C\colon \mathcal{I} \longrightarrow \mathcal{C}$ with colimit $(L, l\colon C \Rightarrow L)$, the geometric axioms

$$\forall x : L \bigvee_{i \in \mathcal{I}} \exists y : C_i \ \ (l_C(y) = x) \text{ and}$$

$$\forall x{:}C \forall y{:}C'(l_C(x) = l_{C'}(y) \longrightarrow \bigvee_{\mathcal{Z}(C,C')} \exists z_1{:}C_1 ... \exists z_n{:}C_n \ (d_{1,0}(z_1) = x \wedge d_{1,1}(z_1) = d_{2,0}(z_2) \wedge ... \wedge d_{n,1}(z_n) = y))$$

where the (infinite) disjunction runs over all the zig-zags

$$
\begin{tikzcd}
& C_1 \arrow[dl, "d_{1,0}"'] \arrow[dr, "d_{1,1}"] & & & C_n \arrow[dl, "d_{n,0}"'] \arrow[dr, "d_{n,1}"] & \\
C & & \cdot \cdots\cdots \cdots\cdots \cdot & & & C'
\end{tikzcd}
$$

from $D(C)$ to $D(C')$.

A general form of Diaconescu's theorem

Theorem: (A. Kock) Let $(\mathcal{K}, j)$ be a cocomplete, finitely complete, subcanonical site and let $F\colon \mathcal{C} \longrightarrow \mathcal{K}$ be a flat functor. Let $D\colon \mathcal{I} \longrightarrow [\mathcal{C}^{op}, \mathsf{Set}]$ be a finite diagram. If the colimits used for constructing $(\mathsf{Lan}_y F)(D_i)$ are postulated, for all $i \in \mathcal{I}$, then $\mathsf{Lan}_y F$ preserves the limit of the diagram D.

Remarks: 1.The proof given by Kock relies on the classical Diaconescu's theorem. We give a direct proof and obtain the classical theorem as a corollary. Our proof is "local". Shows that existence of cones for cospans (and parallel pairs) in the category of elts implies preservation of pb's by $\mathsf{Lan}_y F$ (and equalizers, respectively).

2. $\mathcal{K}$ need not really be cocomplete. E.g extending along inclusion into the exact completion every diagram $y \downarrow D_i \longrightarrow \mathcal{C} \longrightarrow \mathcal{K}$, used in the calculation of $(\mathsf{Lan}_y F)(D_i)$, has a final subcategory which is an equivalence relation.

3. We are particularly interested in the preservation of limits of equalizer diagrams

$$E \longrightarrow \mathsf{colim}_{i<n} yC_i \overset{f}{\underset{g}{\rightrightarrows}} X$$

These in turn are reduced to

$$E \xrightarrow{e} yC \overset{f}{\underset{g}{\rightrightarrows}} \mathsf{colim}_j yC_j$$

(The previous equalizers are retractions of colimits of the above.)

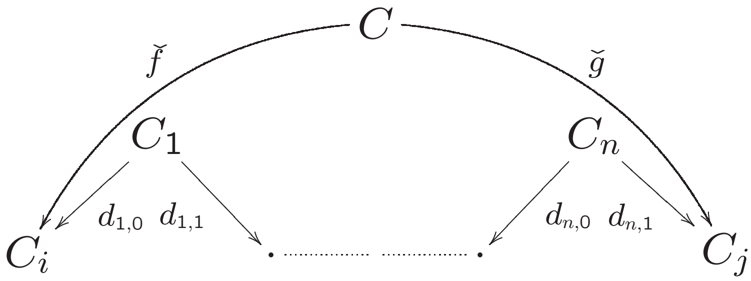
The general result will then follow (in the applications in mind) by the commutation of such equalizers with filtered colimits (which in turn may be a consequence of postulatedness) but doesn't require local cartesian closedness.

A description of equalizers of presheaves

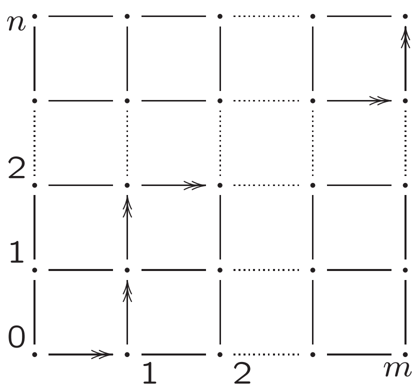
If f and g are represented by arrows $\check{f}\colon C \longrightarrow C_i$ and $\check{g}\colon C \longrightarrow C_j$, respectively, in $\mathcal{C}$, then

$$E \cong \mathrm{colim}_z \mathrm{colim}_k \; yP_{k,z},$$

where $P_{k,z}$ is a final family of cones for the diagram D_z consisting of $\check{f}$, $\check{g}$ and a zig-zag z from C_i to C_j:



Remarks: 1. The final family of cones can be singleton if $\mathcal{C}$ has (weak) limits or finite, as it is the case with $\mathbf{\Delta}$: Given m, n in $\mathbf{\Delta}$ consider the family of pairs of surjective, order-preserving maps $\alpha\colon m+n \longrightarrow m$, $\beta\colon m+n \longrightarrow n$. They correspond to maximal paths on the $m \times n$ grid on the plane, where motion is allowed only upwards and to the right, since α and β are order-preserving:



2. The indexing family of the zig-zags is usually countable but it is finite when $\mathcal{C}$ has BTH of reflexive symmetric relations (as it is the case with $\mathbf{\Delta}$!).

Sketch of Proof:

$\text{colim}_z\text{colim}_k\ FP_{k,z} \longrightarrow Eq(|f|,|g|)$ is "one - one": Given $T-$ elements u, v that get identified in $Eq(|f|,|g|)$ we show that they are equal.

The colimit is postulated so there is $\{t_\alpha\colon T_\alpha \longrightarrow T \mid \alpha \in A\} \in \text{Cov}(T)$ and $FP_{z,k}$, $FP_{z',k'}$ so that, for all $\alpha \in A$, there are factorizations

$$\begin{tikzcd}
& T_\alpha \arrow[dl, "u_\alpha"'] \arrow[d] \arrow[dr, "v_\alpha"] & \\
FP_{z,k} \arrow[dr] & T \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] & FP_{z',k'} \arrow[dl] \\
& \text{colim}_z\text{colim}_k\ FP_{k,z} &
\end{tikzcd}$$

Every T_α is a cone for FD_z (edges λ) and $FD_{z'}$ (edges μ). Thus T_α is a cone for $FD_{z\cup z'}$, the diagram of $F\check{f}$, $F\check{g}$ and the concatenation of Fz, Fz'.

From flatness of F, for all $\alpha \in A$, there are $\{t_{\xi,\alpha}\colon T_{\xi,\alpha} \longrightarrow T_\alpha \mid \xi \in \Xi\}$ such that, for all ξ, there are factorizations

$$\begin{tikzcd}
& & T_{\alpha,\xi} \arrow[d] \arrow[dd, bend left, "w"] & & & \\
& & T_\alpha \arrow[dl, "u_\alpha"'] \arrow[drr, "v_\alpha"] & & & \\
& FP_{z,k} \arrow[d, "\lambda_C"'] \arrow[ddl, bend right] \arrow[ddr, bend left] & FP_{z\cup z',k''} \arrow[l] \arrow[rr] & & FP_{z',k'} \arrow[d, "\mu_C"'] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& FC \arrow[dl, bend right, "F\check{f}"'] \arrow[dr, bend left, "F\check{g}"] & & & FC \arrow[dl, bend right, "F\check{f}"'] \arrow[dr, bend left, "F\check{g}"] & \\
FC_i \arrow[rr, no head, squiggly, "Fz"'] & & FC_j & FC_i \arrow[rr, no head, squiggly, "Fz'"'] & & FC_j
\end{tikzcd}$$

It means that

$$\lambda \circ u_\alpha \circ t_{\xi,\alpha} = \lambda \circ Fp_{k'',k} \circ w$$

and

$$\mu \circ v_\alpha \circ t_{\xi,\alpha} = \mu \circ Fp_{k'',k'} \circ w,$$

i.e we have two factorizations of $T_{\xi,\alpha}$, qua cones for FD_z and $FD_{z'}$, via $FP_{z,k}$ and $FP_{z',k'}$, respectively.

Thus there are compatibility zig-zags z_1, z_2 connecting them through the same cone in the final family:

$$\begin{tikzcd}
 & & & T_{\xi,\alpha} \arrow[dlll, bend right=0, "u_\alpha \circ t_{\xi,\alpha}"'] \arrow[d, "w_{\xi,\alpha,z\cup z',k''}"] \arrow[drrr, "v_\alpha \circ t_{\xi,\alpha}"] & & & \\
 & & & FP_{z\cup z',k''} \arrow[dl, "Fp_{k'',k}"] \arrow[dr, "Fp_{k'',k'}"'] & & & \\
FP_{z,k} \arrow[rr, squiggly, dash, "z_1"'] & & FP_{z,k} & & FP_{z',k'} \arrow[rr, squiggly, dash, "z_2"'] & & FP_{z',k'}
\end{tikzcd}$$

We have found a composite covering

$\{t_\alpha \circ t_{\xi,\alpha} \colon T_{\xi,\alpha} \longrightarrow T \mid (\alpha,\xi) \in A \times \Xi\}$, so that

for all (α,ξ), the $T_{\alpha,\xi}$-elements

$$in_{z,k} \circ u_\alpha \circ t_{\xi,\alpha} = u \circ t_\alpha \circ t_{\xi,\alpha}$$

and

$$in_{z',k'} \circ v_\alpha \circ t_{\xi,\alpha} = v \circ t_\alpha \circ t_{\xi,\alpha}$$

of the colimit become connected via a zig-zag among the components, thus they are equal. Thus $u = v$.

Examples:

1. [KRV], JPAA 196, 229-250, show that when $\mathcal{C}$ has finite final families of cones for finite diagrams and BTH for reflexive symmetric relations then finitely presentable presheaves on it are closed under finite limits. If $\mathcal{C}$ is, in particular, a pretopos with coequalizers and BTH then

$$\mathcal{E} = \mathsf{Lex}(\mathcal{C}^{op}, \mathsf{Set})$$

has postulated finite colimits for the topology of finite regular epi coverings (essentially by older work of Day and Street), while $I \colon \mathcal{C} \longrightarrow \mathcal{E}$ is trivially flat (for all topologies!). Thus the result of Borceux and Pedicchio that $\mathcal{E}$ is a topos follows (the reflexion from presheaves is left exact).

2. Kock shows that if finite (all) colimits are postulated w.r.t a subcanonical topology on $\mathcal{K}$ then $\mathcal{K}$ is a(n ∞-) pretopos. On the other hand Cagliari, Mantovani and Vitale show that the category of Kelley spaces is not exact. So there seems to be no hope for the left exactness of geometric realization of simplicial sets to fit in our explanatory scheme. **But**: By the above remarks on $\mathbf{\Delta}$, the colimits used in the calculation of geometric realization of a finite simplicial set X are of the form

$$\textstyle\coprod_{i<k} \Delta[n_i] \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{j<l} \Delta[m_j] \longrightarrow |X|$$

and such sequences are exact, thus the colimit is postulated.

3. The method appears to be applicable in the case of "connected- limit- flat" functors from finite categories to compact Hausdorff spaces (or other categories...)