ME
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Persistent homology 6½: Categorical approach(es)

[Abstract view of persistent topology... ~~the~~

Like any foundations, seeks to codify what practitioners of the subject do, but inevitably also expresses an opinion on what the subject is about.]

In persistent top.:

- What are the objects of study? [Spaces? Fns? Persistence modules?] return to this
- What is "persistence"?

Ref: Bubenik, de Silva & Scott (2013)

Let X be a MS.

Subspaces	Functions	Persistent spaces	Persistence modules
closed $A \subseteq X$	Cts $X \xrightarrow{f} \mathbb{R}$ E.g. $f = d_A : x \mapsto \inf_a d(a,x)$	Functor $(\mathbb{R}, \leq) \xrightarrow{F} \mathrm{Top}$ E.g. $F(t) = f^{-1}(-\infty, t]$ change of notn wrt Henry	Functor $(\mathbb{R}, \leq) \xrightarrow{\mathcal{F}} \mathrm{Vect}$ E.g. $\mathcal{F}(t) = H_n(F(t))$ (fixed n)
$A \subseteq B_\varepsilon$ $\Leftrightarrow$ (In our example)	$g(x) \leq f(x) + \varepsilon$ $\forall x \in X$ $\Leftrightarrow$	$G(t) \subseteq F(t+\varepsilon)$ $\forall t \in \mathbb{R}$ $\mapsto$	map $\mathcal{G}(t) \longrightarrow \mathcal{F}(t+\varepsilon)$ $\forall t$
Hausdorff dist: $d_H(A,B) =$ $\inf\{\varepsilon : A \subseteq B_\varepsilon, B \subseteq A_\varepsilon\}$	Sup/uniform/L^∞ distance: $\lVert f-g \rVert_\infty =$ $\inf\{\varepsilon : f \leq g+\varepsilon, g \leq f+\varepsilon\}$	Interleaving distance: $d_I(F,G) =$ ~~$\inf\{\varepsilon : F(t) \subseteq G(t+\varepsilon), G(t) \subseteq F(t+\varepsilon)\}$~~ $\inf\{\ldots\}$ same as this	Interleaving distance: $d_I(\mathcal{F},\mathcal{G}) =$ $\inf\{\varepsilon : \exists$ maps $\mathcal{F}(t) \xrightarrow{\varphi_t} \mathcal{G}(t+\varepsilon)$, $\mathcal{G}(t) \xrightarrow{\psi_t} \mathcal{F}(t+\varepsilon)$ $\forall t \in \mathbb{R}$, s.t. some diagrams commute$\}$

In our examples:

$$d_H(A,B) = \lVert f-g \rVert_\infty \geq d_I(F,G) \geq d_I(\mathcal{F},\mathcal{G})$$

[But might not start with a subsp. Might start with a fn, or persistent space, or pers. module]

Abstract approach of B, dS & S :

Let $\mathbb{P}$ be a poset with ~~some~~ certain extra structure (a metric will do). E.g. $(0<1<\dots<n)$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^m$.

[Reflects time, or scale, or persistence.]

A <u>(generalised) persistence module</u> is a functor from $\mathbb{P}$ into any other category $\mathbb{D}$ (e.g. Simp, Top, Vect, Set).

There's a metric ("interleaving distance") on $\mathbb{D}^{\mathbb{P}} = \{$functors $\mathbb{P} \to \mathbb{D}\}$. ~~This functorial~~: For any functor $\mathbb{D} \xrightarrow{H} \mathbb{E}$, the induced functor

$$\mathbb{D}^{\mathbb{P}} \xrightarrow{H_*} \mathbb{E}^{\mathbb{P}}, \quad F \longmapsto H \circ F$$

is distance-dec.

i.e. for $\mathbb{P} \overset{F}{\underset{G}{\rightrightarrows}} \mathbb{D}$, $d(H\circ F, H \circ G) \le d(F, G)$.

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~~Stability~~

$$\overbrace{d_{\text{bottleneck}}(\text{pers. diagm of } f, \text{ of } g)}^{\text{last}} \le d_I(H_n F, H_n G) \underset{\text{as in B, dS \& S}}{\le} d_I(F, G) = d_\infty(f, g) = d_H(A, B).$$

"hard" [in both senses]

[uses technical hyps on triangulability & dimension]

<u>Conclusion</u>: if $A, B \subseteq X$ are Hausdorff-close, their pers. diagrams are close.