THE CARDINALITY OF A METRIC SPACE

Tom Leinster (Glasgow/EPSRC)

Partly joint with Simon Willerton (Sheffield)

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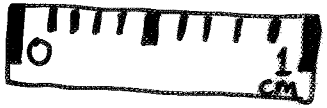
Contents

1. SOME GEOMETRIC MEASURE THEORY

Ref: Schanuel, "What is the length of a potato?"

Suppose we want a ruler of length 1 cm: 

A half-open interval is good:

A closed interval is not so good:

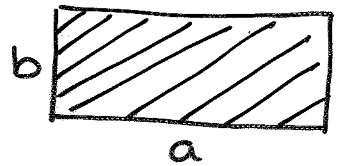
So we declare

$$\text{measure}([0,1]) = 1\,\text{cm} + 1\,\text{point} = 1\,\text{cm}^1 + 1\,\text{cm}^0 = 1\,\text{cm} + 1.$$

Similarly,

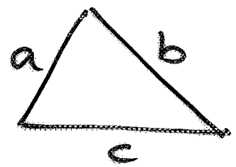
$$\boxed{\text{measure}([0,a]) = a\,\text{cm} + 1.}$$

Take this idea seriously

E.g.: • Measure of rectangle 

is

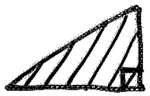
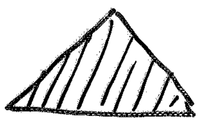
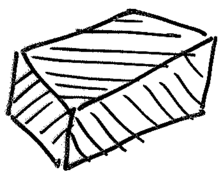
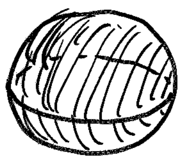
$$(a\ \mathrm{cm}+1)(b\ \mathrm{cm}+1) = \underbrace{ab}_{\text{area}}\ \mathrm{cm}^2 + \underbrace{(a+b)}_{\frac{1}{2}\cdot\text{perimeter}}\mathrm{cm} + \underbrace{1}_{\text{Euler characteristic}}.$$

• Measure of a <u>hollow</u> triangle 

is

$$(a\ \mathrm{cm}+1) + (b\ \mathrm{cm}+1) + (c\ \mathrm{cm}+1) - 3$$
$$= (a+b+c)\ \mathrm{cm}. + 0$$

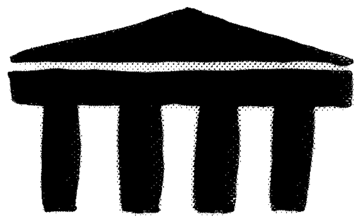
• Similarly, can calculate measures of

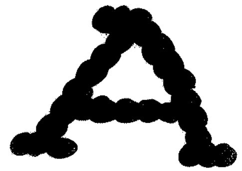
 ,  ,  ,  , ...

Hadwiger's Theorem

(Ref: Klain & Rota, "Intro to Geometric Probability")

Let $n \in \mathbb{N}$. A subset of $\mathbb{R}^n$ is polyconvex if it is a finite union of compact convex sets.





A "measure" is a function

$$|\cdot| : \{\text{polyconvex subsets of } \mathbb{R}^n\} \longrightarrow \mathbb{R}$$

satisfying some axioms, including

$$|A \cup B| = |A| + |B| - |A \cap B|, \qquad |\emptyset| = 0$$

(but not necessarily <u>infinitely</u> additive).

The measures form a real vector space, V_n.

(Reminder: V_n is vector space of measures on {polyconvex subsets of $\mathbb{R}^n$}.)

Thm (Hadwiger): $\dim V_n = n+1$.

There is a basis

$$|\cdot|_0, \ |\cdot|_1, \ \ldots, \ |\cdot|_n$$

where $|\cdot|_d$ is a "d-dimensional" measure.

E.g.:
- V_2 has a basis
  Euler characteristic, perimeter, area.
- For all n,
  $|\cdot|_0$ = Euler characteristic,
  $|\cdot|_n$ = Lebesgue measure on $\mathbb{R}^n$.

2. THE CARDINALITY OF A CATEGORY

(EULER CHARACTERISTIC)

Ref: math. CT/0610260

To each finite set X there is assigned a number $|X|$ (its cardinality).

Let $\mathbb{A}$ be a finite category with objects $a_1, \ldots, a_n$.

Let Z be the $n \times n$ matrix $\left(|\mathrm{Hom}(a_i, a_j)|\right)_{i,j}$. Assume invertible.

The cardinality of $\mathbb{A}$ is $|\mathbb{A}| = \sum_{i,j} (Z^{-1})_{i,j} \in \mathbb{Q}$.

E.g.:

- $\mathbb{A} = (\bullet \rightrightarrows \bullet)$: $Z = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $Z^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, $|\mathbb{A}| = 1 - 2 + 0 + 1 = 0 \ (= \chi(S^1))$
- $\mathbb{A}$ = discrete cat on n objects: $Z = I_n = Z^{-1}$, $|\mathbb{A}| = n$

We have $|\mathbb{A} \times \mathbb{B}| = |\mathbb{A}| \cdot |\mathbb{B}|$ for categories $\mathbb{A}$ and $\mathbb{B}$,

using fact that $|X \times Y| = |X| \cdot |Y|$ for sets X and Y.

2. THE CARDINALITY OF AN ENRICHED CATEGORY

Let $\mathcal{V}$ be a monoidal category. Suppose that to each $\mathcal{V}$-object X there is assigned a number $|X|$ (its cardinality).

Let $\mathbb{A}$ be a $\mathcal{V}$-category with objects $a_1, \ldots, a_n$.

Let Z be the $n \times n$ matrix $\left(|\mathrm{Hom}(a_i, a_j)|\right)_{i,j}$. Assume invertible.

The cardinality of $\mathbb{A}$ is $|\mathbb{A}| = \sum_{i,j} (Z^{-1})_{i,j}$.

E.g.: $\mathcal{V}$ = finite-dimensional vector spaces, usual $\otimes$, $|X| = \dim X$

We have $|\mathbb{A} \otimes \mathbb{B}| = |\mathbb{A}| \cdot |\mathbb{B}|$ for $\mathcal{V}$-categories $\mathbb{A}$ and $\mathbb{B}$,

as long as $|X \otimes Y| = |X| \cdot |Y|$ for $\mathcal{V}$-objects X and Y.

3. METRIC SPACES AS ENRICHED CATEGORIES

(Ref: Lawvere, "Metric spaces...", Reprints in TAC)

Let $\mathcal{V} = ([0,\infty], \geqslant)$: poset, hence category.

(One arrow $x \to y$ if $x \geqslant y$; no arrows $x \to y$ if $x < y$.)

Monoidal category under $(+, 0)$.

A $\mathcal{V}$-category consists of:

- a set A of "objects" or "points"
- for each $a, b \in A$, a number $\mathrm{Hom}(a,b) = d(a,b) \in [0,\infty]$
- for each $a, b, c \in A$, an inequality $d(a,b) + d(b,c) \geqslant d(a,c)$
- for each $a \in A$, an inequality $0 \geqslant d(a,a)$ $(\Leftrightarrow d(a,a) = 0)$.

So any metric space is a $\mathcal{V}$-category.

Review of talk so far

Schanuel: $[0, a]$ should have "measure" $(a\,\mathrm{cm} + 1)$

Hadwiger: on polyconvex subsets of $\mathbb{R}^n$, there are

$(n+1)$ "measures" $|\cdot|_0, |\cdot|_1, \ldots, |\cdot|_n$

Cardinality of a finite cat $\mathbb{A}$: let $Z_{ij} = |\mathrm{Hom}(a_i, a_j)|$;

then $|\mathbb{A}| = \sum_{i,j} (Z^{-1})_{ij}$

Cardinality of an enriched cat: very similar

Lawvere: metric spaces are enriched cats.

4. THE CARDINALITY OF A FINITE METRIC SPACE

(Ref: "Metric spaces", post at n-Category Café, 9 Feb 2008)

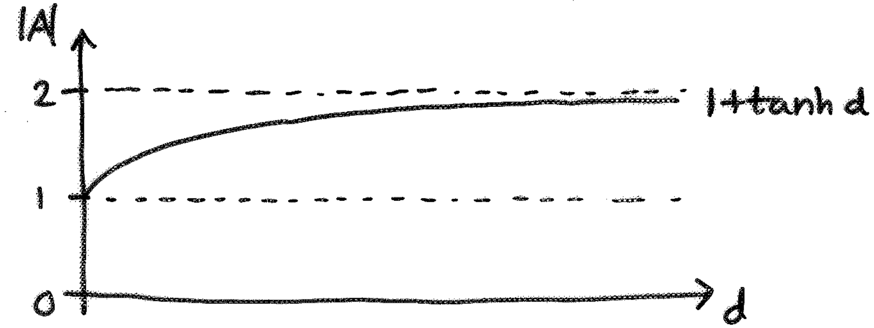
For $x \in [0, \infty]$, define $|x| = e^{-2x}$. Then $|x+y| = |x| \cdot |y|$.

Defn: Let $A = \{a_1, \ldots, a_n\}$ be a finite metric space.

Define an $n \times n$ matrix Z by $Z_{ij} = e^{-2d(a_i, a_j)}$.

The *cardinality* of A is $|A| = \sum_{i,j} (Z^{-1})_{ij} \in \mathbb{R}$.

E.g.: $A = \left(\overset{a_1}{\bullet} \xleftrightarrow{\;d\;} \overset{a_2}{\bullet} \right)$. Then $Z = \begin{pmatrix} 1 & e^{-2d} \\ e^{-2d} & 1 \end{pmatrix}$ and $|A| = \dfrac{2}{1+e^{-d}} = 1 + \tanh(d)$.



Warning (Tao): There exist finite metric spaces whose cardinality is undefined (Z is not invertible).

Defn: Let A be a finite metric space.

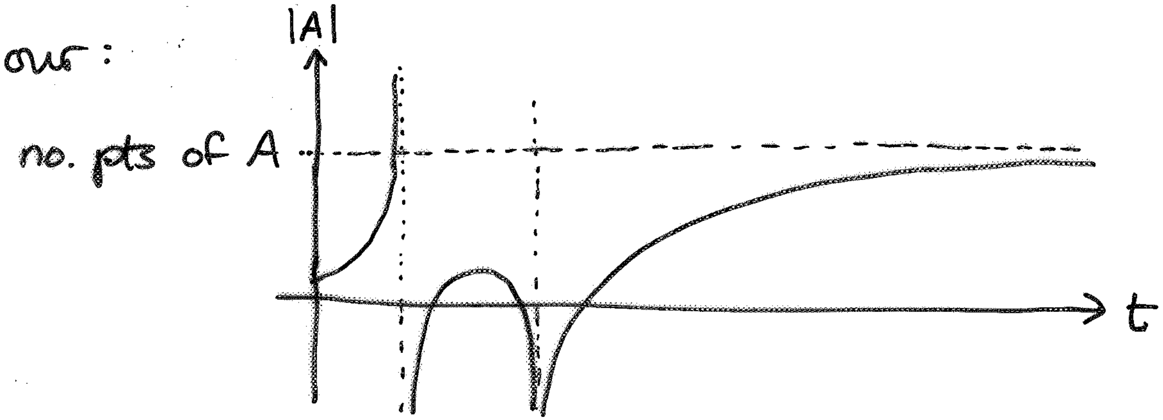
The cardinality function of A is the partial function

$$\chi_A : (0,\infty) \longrightarrow \mathbb{R}$$
$$t \longmapsto |tA|.$$

A scaled up by a factor of t

Eg: $\chi_A(1) = |A|$.

Typical behaviour:



5. THE CARDINALITY OF A COMPACT METRIC SPACE

Idea: Given a compact metric space A, take a sequence

$$A_0 \subseteq A_1 \subseteq \cdots$$

of finite subsets of A, with $\bigcup_n A_n$ dense in A, and try to define $|A| = \lim_{n\to\infty} |A_n|$.

Thm: Let $A = [0, a]$ and take $(A_n)_{n\geq 0}$ as above. Then

$$\lim_{n\to\infty} |A_n| = a + 1.$$

Remarks:

- This is the reason for the choice of e^{-2} (otherwise get $(\text{constant} \cdot a + 1)$ on RHS)
- $\chi_{[0,a]}(t) = |[0, ta]| = at + 1$: "$t = $ cm".

Assume now that all spaces mentioned have well-defined cardinality...

Example (Products): Let A, B be compact metric spaces. Then $|A \times B| = |A| \cdot |B|$ <u>if</u> we give $A \times B$ the "d_1 metric":

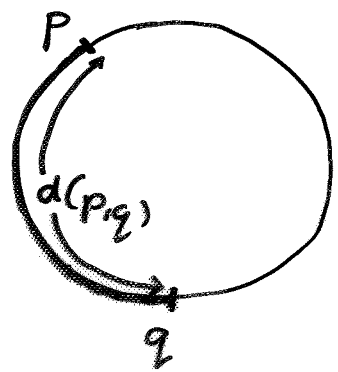
$$d((a,b), (a',b')) = d(a,a') + d(b,b').$$

With this metric, $[0,a] \times [0,b]$ has cardinality function

$$t \longmapsto (at+1)(bt+1)$$
$$= ab\,t^2 + (a+b)t + 1.$$

Example (circles):

Let C_a be the circle of <u>circumference</u> a, where metric is given by length of shortest arc.



Then

$$|C_a| = \frac{a}{1-e^{-a}} = \sum_{n \geq 0} B_n \frac{(-a)^n}{n!}$$

Bernoulli numbers

So $|C_a| \longrightarrow 1$ as $a \to 0$ and $|C_a| \sim a$ $(= a + 0\ !)$ as $a \to \infty$.

Conjectures

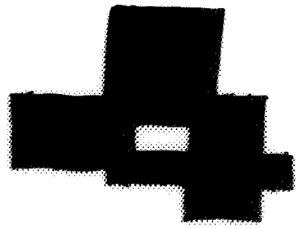
Conjecture (confident):

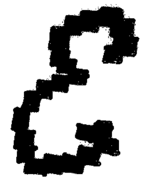
Let A be a finite, connected union of cuboids

$$[r_1, s_1] \times \cdots \times [r_n, s_n]$$

in $\mathbb{R}^n$. Give $A \subseteq \mathbb{R}^n$ the d_1 metric. Then

$$\chi_A(t) = |A|_n t^n + \cdots + |A|_1 t + |A|_0 .$$





Conjecture (guess):

Let A be a polyconvex subset of $\mathbb{R}^n$. Then

$$\chi_A(t) \sim |A|_n t^n + \cdots + |A|_1 t + |A|_0$$

(i.e. LHS − RHS $\to 0$ as $t \to \infty$.)

Both conjectures say:

cardinality includes all the invariants $|\cdot|_d$ of geometric measure theory.