Magnitude homology equivalence of Euclidean sets

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Paper in preparation with
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These slides: on my web page

Summary

Magnitude homology is a homology theory of enriched categories.

It specializes to a homology theory of metric spaces.

Main theorem (with Adrián Doña Mateo) *Two closed subsets of $\mathbb{R}^N$ are magnitude homology equivalent if and only if they are related by a certain concrete geometric condition.*

Motivation

In topology, we have the concept of two continuous maps

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

being *homotopic*.

We also know that *homotopic maps induce the same map* $H_*(X) \to H_*(Y)$ in homology.

Question Is there an analogous concept of homotopy for magnitude homology of metric spaces, with an analogous theorem?

Yu Tajima and Masahiko Yoshinaga have looked at this question in one way.

We look at it in another.

Plan

1. What is magnitude homology?
2. Preparation for the main theorem
3. The main theorem
4. What goes into the proof?

1. What is magnitude homology?

Richard Hepworth and Simon Willerton, Categorifying the magnitude of a graph

Tom Leinster and Michael Shulman, Magnitude homology of enriched categories and metric spaces

Warm-up: homology of an ordinary category

Any *ordinary* category $\boldsymbol{X}$ gives rise to a chain complex $C_*(\boldsymbol{X})$:

$$C_n(\boldsymbol{X}) = \coprod_{x_0,\ldots,x_n \in \boldsymbol{X}} \mathbb{Z} \cdot \big(\boldsymbol{X}(x_0, x_1) \times \cdots \times \boldsymbol{X}(x_{n-1}, x_n)\big)$$

where $\mathbb{Z} \cdot -$: **Set** $\to$ **Ab** is the free abelian group functor.

The differential ∂ is $\sum_{i=0}^{n} (-1)^i \partial_i$, where ∂_i composes at x_i (for $0 < i < n$) or forgets the first/last factor (for $i \in \{0, n\}$).

The homology $H_*(\boldsymbol{X})$ of $\boldsymbol{X}$ is the homology of $C_*(\boldsymbol{X})$.

Key ingredients here:

- $(\mathbf{Set}, \times, 1)$ is a monoidal category, whose unit object 1 is terminal.
- **Ab** is both abelian and monoidal.
- $\mathbb{Z} \cdot -$ is a strong monoidal functor ($\mathbb{Z} \cdot (S \times T) \cong \mathbb{Z} \cdot S \otimes \mathbb{Z} \cdot T$).

The magnitude homology of an enriched category: setup

Imitating the ordinary, unenriched case, we start with:

- a monoidal category $\boldsymbol{V}$ whose unit object is terminal (generalizing **Set**)
- a monoidal abelian category $\boldsymbol{A}$ (generalizing **Ab**)
- a strong monoidal functor $\Sigma\colon \boldsymbol{V} \to \boldsymbol{A}$ (generalizing $\mathbb{Z} \cdot -$).

Analogy This is a categorification of the setup for magnitude, which is:

- a monoidal category $\boldsymbol{V}$
- a semiring k
- a monoid homomorphism $|\cdot|\colon (\mathrm{ob}\,\boldsymbol{V})/\cong \to k$.

The magnitude homology of an enriched category: definition

We start with:

- a monoidal category $\boldsymbol{V}$ whose unit object is terminal
- a monoidal abelian category $\boldsymbol{A}$
- a strong monoidal functor $\Sigma\colon \boldsymbol{V} \to \boldsymbol{A}$.

Let $\boldsymbol{X}$ be a category enriched in $\boldsymbol{V}$.

Define a chain complex $C_*(\boldsymbol{X})$ in $\boldsymbol{A}$ by

$$C_n(\boldsymbol{X}) = \coprod_{x_0,\ldots,x_n \in \boldsymbol{X}} \Sigma\big(\boldsymbol{X}(x_0, x_1) \otimes \cdots \otimes \boldsymbol{X}(x_{n-1}, x_n)\big).$$

It has differential $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$, where ∂_i either composes at x_i or forgets the first/last factor.

The magnitude homology $MH_*(\boldsymbol{X})$ of $\boldsymbol{X}$ is the homology of $C_*(\boldsymbol{X})$.

The magnitude homology of a metric space

Metric spaces are categories enriched in $\boldsymbol{V} = (([0,\infty), \geq), +, 0)$.

To take the magnitude homology of metric spaces, we'll need:

- a monoidal abelian category $\boldsymbol{A}$
- a strong monoidal functor $\Sigma\colon [0,\infty) \to \boldsymbol{A}$.

We choose:

- $\boldsymbol{A} = \mathbf{Ab}^{[0,\infty)}$ with the convolution product: $(A \otimes B)_\ell = \coprod_{k+m=\ell} A_k \otimes B_m$
- $\Sigma\colon [0,\infty) \to \mathbf{Ab}^{[0,\infty)}$ to be the functor defined by

$$(\Sigma(\ell))(m) = \begin{cases} \mathbb{Z} & \text{if } \ell = m \\ 0 & \text{otherwise} \end{cases}$$

$(\ell, m \in [0,\infty))$.

The magnitude homology of a metric space, explicitly

Let X be a metric space.

The chain complex $C_{*,*}(X)$ in $\mathbf{Ab}^{[0,\infty)}$ is given by

$$C_{n,\ell}(X) = \mathbb{Z} \cdot \big\{(x_0, \ldots, x_n) \ : \ d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell\big\}$$

$(n \in \mathbb{N}, \ell \in [0, \infty))$.

Equivalently, we can replace $C_{*,*}(X)$ by a normalized version, $\hat{C}_{*,*}(X)$:

$$\hat{C}_{n,\ell}(X) = \mathbb{Z} \cdot \big\{(x_0, \ldots, x_n) \ : \ x_0 \neq \cdots \neq x_n, \ d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell\big\}.$$

The differential $\partial \colon \hat{C}_n(X) \to \hat{C}_{n-1}(X)$ is $\sum_{0<i<n} (-1)^i \partial_i$, where

$$\partial_i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{if } x_i \text{ is between } x_{i-1} \text{ and } x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $MH_{*,*}(X)$ is the homology of the chain complex $\hat{C}_{*,*}(X)$ in $\mathbf{Ab}^{[0,\infty)}$.

Magnitude homology is graded!

Magnitude homology of a metric space is a $[0, \infty)$-*graded* homology theory.

That is, when X is a metric space and n is a natural number, $MH_n(X)$ is not just an abelian group, but an object of $\mathbf{Ab}^{[0,\infty)}$— a *family*

$$\big(MH_{n,\ell}(X)\big)_{\ell \in [0,\infty)}$$

of abelian groups.

(Compare Khovanov homology. . .)

Sample results

- Let X be a closed subset of $\mathbb{R}^N$. Then

$$X \text{ is convex} \iff MH_{1,\ell}(X) = 0 \text{ for all } \ell > 0.$$

In fact, if X is convex then $MH_{n,\ell}(X) = 0$ for all $n \geq 1$ and $\ell > 0$.

- Work of Kyonori Gomi and others gives evidence for the slogan:

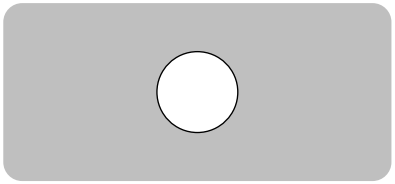
  The more geodesics are unique, the more magnitude homology is trivial.

- Ordinary homology detects the *existence* of holes.

Magnitude homology detects the *size* of holes.

Example (Ryuki Kaneta & Masahiko Yoshinaga) Let $r > 0$ and

$$X = \{x \in \mathbb{R}^N \ : \ \|x\| \geq r\}.$$



Then $r = \sup\{\ell/2n \ : \ MH_{n,\ell}(X) \neq 0\}$.

2. Preparation for the main theorem

What does it mean to "have the same homology"?

For *any* homology theory, what does it mean for two objects X and Y to "have the same homology"? There are several interpretations...

Answer 1 *Crude:* $H_*(X) \cong H_*(Y)$.

Usually seen as unhelpful, too loose.

Unhelpful for us too. E.g. Emily Roff has exhibited metric spaces with the same magnitude homology (in this sense) but different topological homology.

Answer 2 *Quasi-isomorphism:* Declare X and Y to "have the same homology" if there is a map $X \to Y$ inducing an iso $H_*(X) \to H_*(Y)$.

Answer 3 *One step further:* Ask for existence of maps $X \rightleftarrows Y$ inducing mutually inverse maps $H_*(X) \rightleftarrows H_*(Y)$.

We follow Answer 3, where our objects are metric spaces and map means distance-decreasing (= 1-Lipschitz = weakly contractive = short) map: $d(f(x), f(x')) \leq d(x, x')$.

Another preview of the main theorem

Theorem (with Adrián Doña Mateo) *Let X and Y be nonempty closed subsets of $\mathbb{R}^N$. The following are equivalent:*

- *there are distance-decreasing maps $X \rightleftarrows Y$ inducing mutually inverse maps $MH_{n,*}(X) \rightleftarrows MH_{n,*}(Y)$ for all $n \geq 1$*
- *there are distance-decreasing maps $X \rightleftarrows Y$ inducing mutually inverse maps $MH_{n,*}(X) \rightleftarrows MH_{n,*}(Y)$ for some $n \geq 1$*
- *X and Y are related by a certain concrete geometric condition.*

Next: that "concrete geometric condition".

The inner boundary of a space

Let X be a metric space.

Points $x, y \in X$ are adjacent if they are distinct and there is no point $z \in X$ strictly between them: $d(x,z) + d(z,y) = d(x,y) \Rightarrow z \in \{x, y\}$.

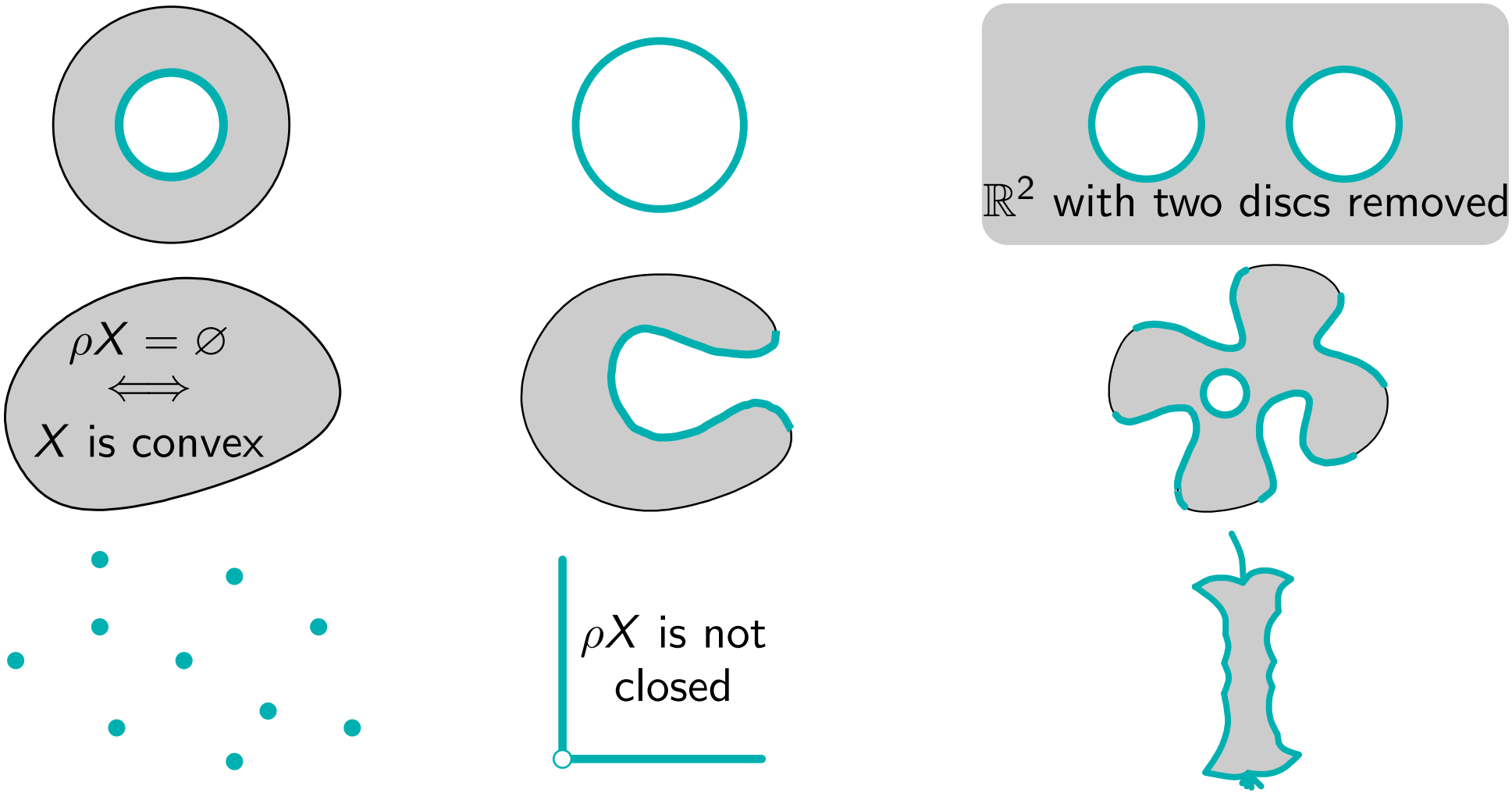
The inner boundary of X is

$$\rho X = \{x \in X \ : \ x \text{ is adjacent to some point of } X\}.$$

Note When $X \subseteq \mathbb{R}^N$, the inner boundary is a subset of the topological boundary: $\rho X \subseteq \partial X$.

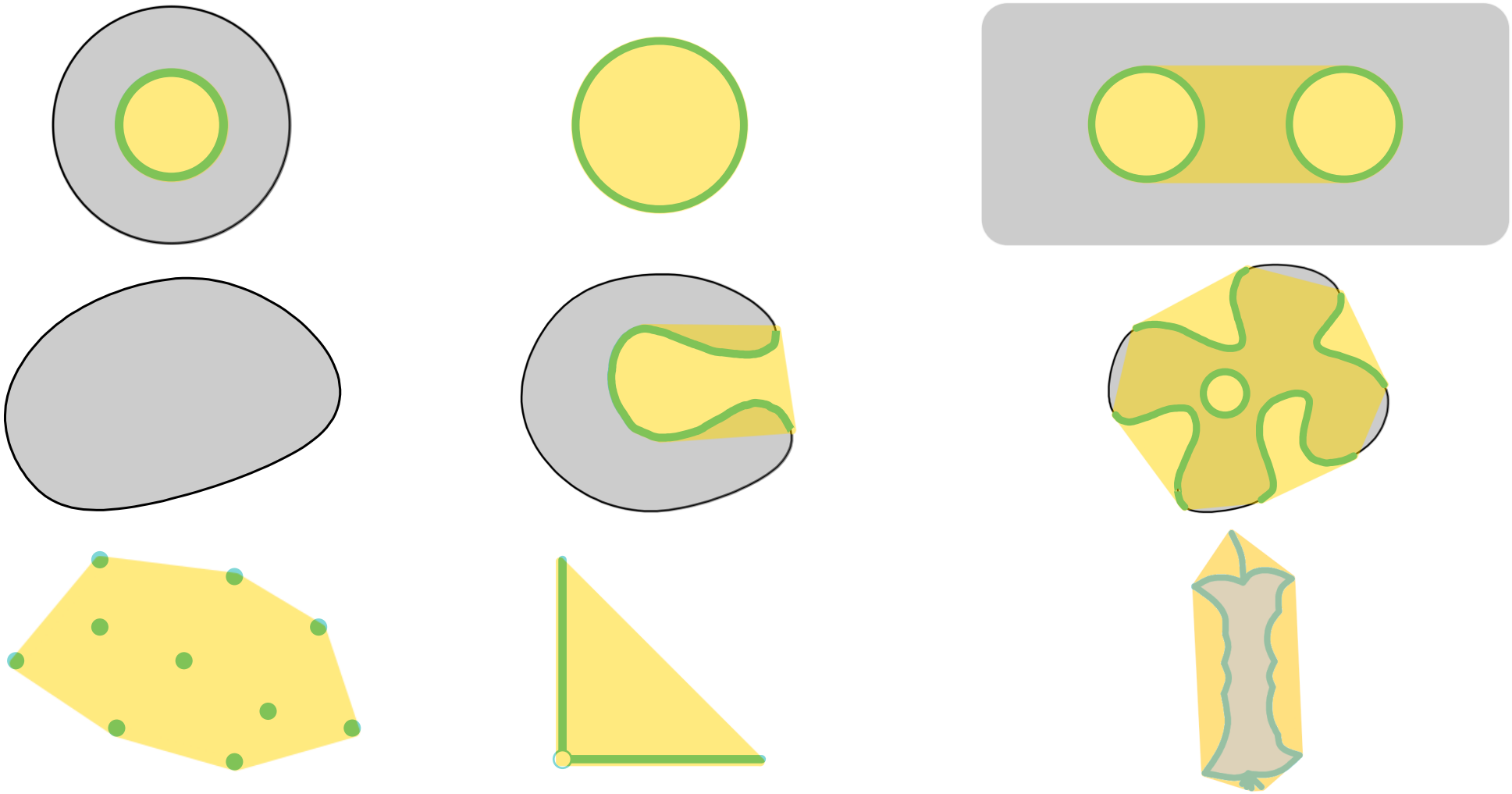
Examples of inner boundaries (all closed subsets of $\mathbb{R}^N$)

- ρX: inner boundary of X (the set of points adjacent to some other point)



The core of a subset of $\mathbb{R}^N$

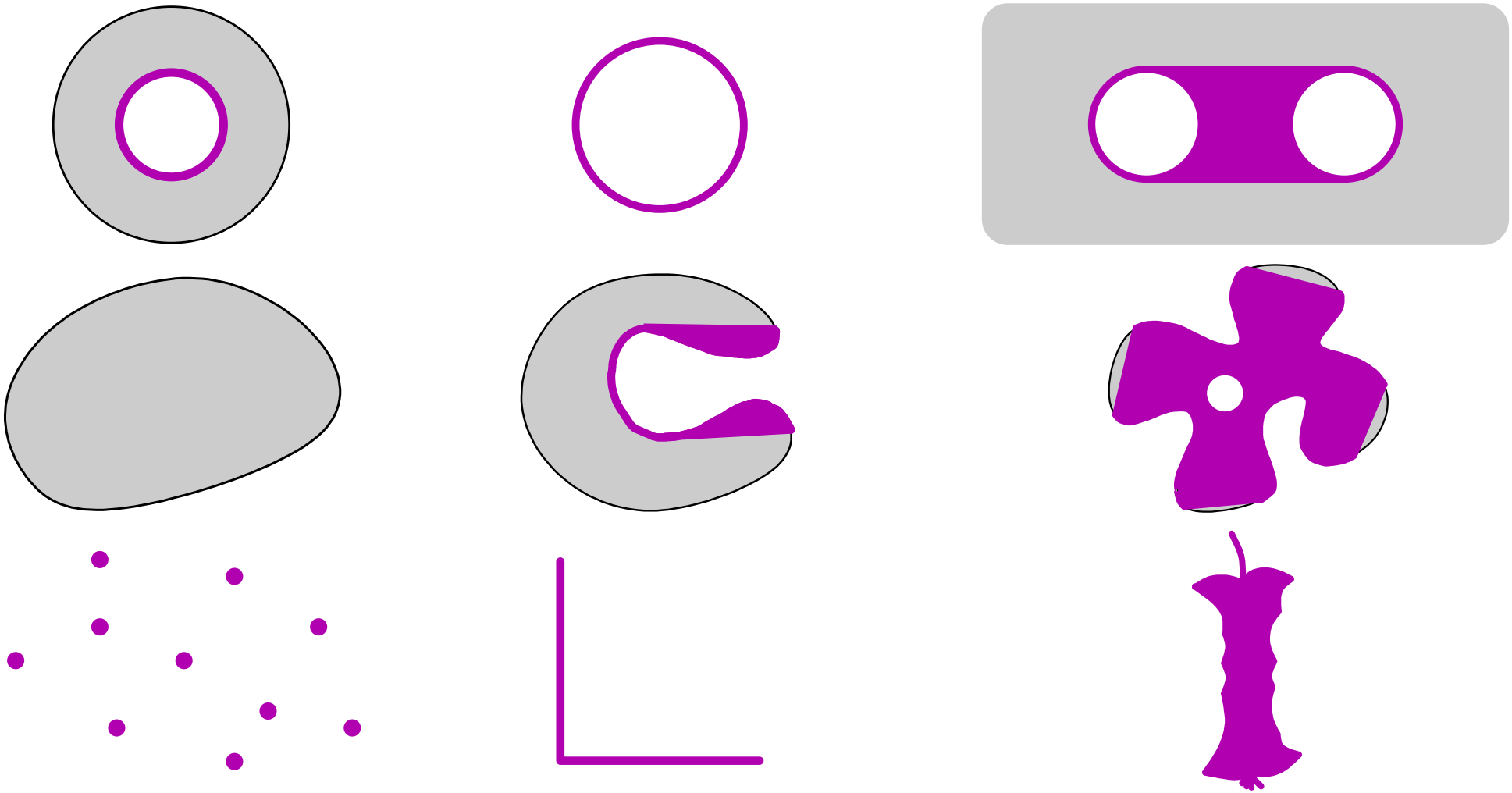
- ρX: inner boundary of X (the set of points adjacent to some other point)
- $\overline{\mathrm{conv}(\rho X)}$: closure of convex hull of ρX



The core of a subset of $\mathbb{R}^N$

- ρX: inner boundary of X (the set of points adjacent to some other point)
- $\overline{\mathrm{conv}(\rho X)}$: closure of convex hull of ρX
- $\mathrm{core}(X) = \overline{\mathrm{conv}(\rho X)} \cap X$

Fact: $\mathrm{core}(\mathrm{core}(X)) = \mathrm{core}(X)$



3. The main theorem

The main theorem

Theorem (with Adrián Doña Mateo) *Let X and Y be nonempty closed subsets of $\mathbb{R}^N$. The following are equivalent:*

(i) *there exist distance-decreasing maps $X \rightleftarrows Y$ inducing mutually inverse maps $MH_{n,*}(X) \rightleftarrows MH_{n,*}(Y)$ for* all *$n \geq 1$*

(ii) *there exist distance-decreasing maps $X \rightleftarrows Y$ inducing mutually inverse maps $MH_{n,*}(X) \rightleftarrows MH_{n,*}(Y)$ for* some *$n \geq 1$*

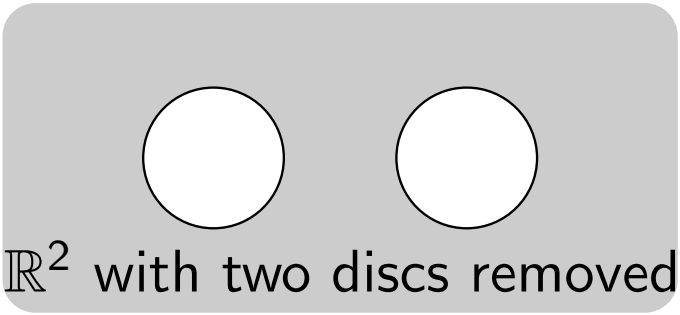
(iii) $\mathrm{core}(X)$ *and* $\mathrm{core}(Y)$ *are isometric.*

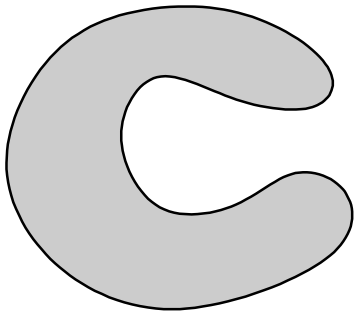
In particular, X and $\mathrm{core}(X)$ have the same magnitude homology, for any X.

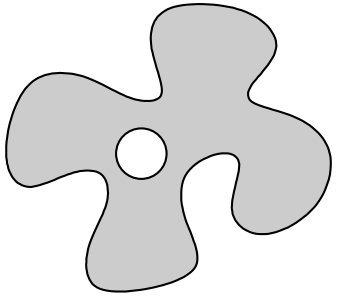
Magnitude homology equivalence
reduces to a concrete geometric condition,
for closed subsets of Euclidean space.

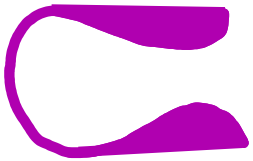
Examples

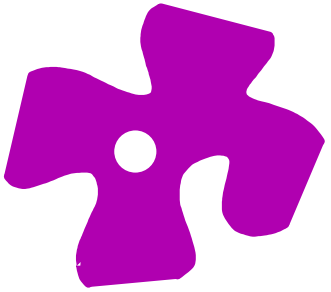
Each of these pairs has the same magnitude homology in degree ≥ 1:











4. What goes into the proof?

The ingredients, in brief

- Kaneta and Yoshinaga's structure theorem for magnitude homology.
- An analysis of when two maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ of metric spaces induce the same map in magnitude homology.
- Some convex geometry.

Now for some more detail...

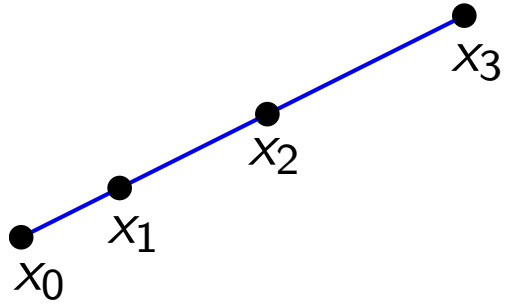
Straight metric spaces

Let X be a metric space. For $x, y \in X$, define

$$\begin{aligned}[x, y] &= \{z \in X \ : \ z \text{ is between } x \text{ and } y\} \\ &= \{z \in X \ : \ d(x, z) + d(z, y) = d(x, y)\}.\end{aligned}$$

Call X straight if whenever $x_0 \neq x_1 \neq \cdots \neq x_n$ with $x_i \in [x_{i-1}, x_{i+1}]$ for all i, we have

$$[x_0, x_n] = [x_0, x_1] \cup \cdots \cup [x_{n-1}, x_n].$$



Examples

- $\mathbb{R}^N$ is straight.
- Any subspace of a straight space is straight.

Lemma *Straight* $\iff$ *geodetic and no 4-cuts.*

The definitions of 'geodetic' and 'no 4-cuts' won't be needed today.

Informally "Straight" means that the *betweenness* relation behaves as in subsets of $\mathbb{R}^N$.

Kaneta and Yoshinaga's structure theorem

By definition, an element of $MH_{n,*}(X)$ is an equivalence class of cycles $\mathbf{x} = (x_0, \ldots, x_n)$.

An $(n+1)$-tuple of points $\mathbf{x} = (x_0, \ldots, x_n)$ is thin if:

- x_{i-1} and x_i are adjacent for all i (no point of X is between them)
- x_i is not between x_{i-1} and x_{i+1}, for any i.

Then every x_i is in ρX, and $\mathbf{x}$ is automatically a cycle.

Theorem (Kaneta and Yoshinaga) *If X is straight then $MH_{n,*}(X)$ is freely generated by the set of thin $(n+1)$-tuples.*

In fact, they proved something more precise.

But even this crude version has important consequences, e.g.:

- the magnitude homology of a straight space only depends on its inner boundary
- convex subsets of $\mathbb{R}^N$ have trivial magnitude homology in degree ≥ 1.

When are two maps the same in homology?

For us, a map of metric spaces is a (non-strictly) distance-decreasing map.

Theorem *Take maps of metric spaces* $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, *with* X *straight. If*

$f|_{\rho X} = g|_{\rho X}$ *then* $C_{*,*}(X) \underset{g_\#}{\overset{f_\#}{\rightrightarrows}} C_{*,*}(Y)$ *are chain homotopic in degree* ≥ 1.

Hence $MH_{*,*}(X) \underset{g_*}{\overset{f_*}{\rightrightarrows}} MH_{*,*}(Y)$ *are equal.*

Proof 1 Construct an explicit chain homotopy.

Proof 2 Follows from Kaneta and Yoshinaga's structure theorem by a homological algebraic argument.

Corollary *Let* X *be a straight metric space. Then any retract of* X *containing its inner boundary has the same homology as* X.

When does a self-map induce the identity in homology?

Theorem *Let X be a straight metric space. The following conditions on a self-map $e\colon X \to X$ are equivalent:*

- $e_*\colon MH_{n,*}(X) \to MH_{n,*}(X)$ *is the identity for* all $n \geq 1$
- $e_*\colon MH_{n,*}(X) \to MH_{n,*}(X)$ *is the identity for* some $n \geq 1$
- e *is the identity on the inner boundary of* X.

The main ingredient of the proof is Kaneta and Yoshinaga's structure theorem, again.

All $\iff$ some???

This may seem surprising at first, but...

Key point The magnitude homology of straight spaces is rather simple, because:

- betweenness behaves like in subsets of $\mathbb{R}^N$;
- geodesics in $\mathbb{R}^N$ are straightforward;
- "the more geodesics are unique, the more magnitude is trivial" (Gomi).

The convex geometry part

Our main theorem is about subsets of $\mathbb{R}^N$, not arbitrary straight spaces.

What specific properties of $\mathbb{R}^N$ do we use?

Some of them:

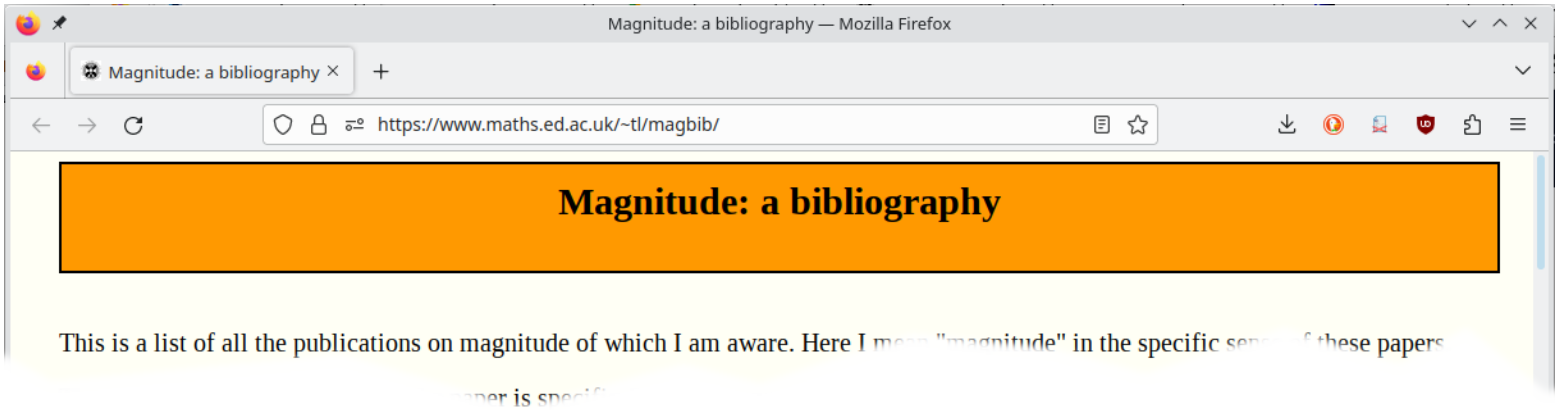
- every interval $[x, y]$ is compact
- every isometry from a subset of $\mathbb{R}^N$ to $\mathbb{R}^N$ extends to an isometric isomorphism $\mathbb{R}^N \to \mathbb{R}^N$
- for closed $X \subseteq \mathbb{R}^N$, we have $\mathrm{conv}(X) = X \cup \mathrm{conv}(\rho X)$, where conv means convex hull.

References

Ryuki Kaneta and Masahiko Yoshinaga, Magnitude homology of metric spaces and order complexes.

Yu Tajima and Masahiko Yoshinaga, Causal order complex and magnitude homotopy type of metric spaces.

The magnitude bibliography:



... plus paper with Adrián Doña Mateo in preparation.