The many faces of magnitude

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Plan

1. Magnitude: the big picture

2. The magnitude of a metric space

$2\frac{1}{2}$. The magnitude of a graph

3. Magnitude homology

4. (Bio)diversity

1. Magnitude: the big picture

The idea

For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies

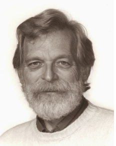
$$\begin{aligned} |S \cup T| &= |S| + |T| - |S \cap T| \\ |S \times T| &= |S| \times |T|. \end{aligned}$$

- Subsets of $\mathbb{R}^n$ have volume. It satisfies

$$\begin{aligned} \mathrm{vol}(S \cup T) &= \mathrm{vol}(S) + \mathrm{vol}(T) - \mathrm{vol}(S \cap T) \\ \mathrm{vol}(S \times T) &= \mathrm{vol}(S) \times \mathrm{vol}(T). \end{aligned}$$

- Topological spaces have Euler characteristic. It satisfies

$$\begin{aligned} \chi(S \cup T) &= \chi(S) + \chi(T) - \chi(S \cap T) \quad \text{(under hypotheses)} \\ \chi(S \times T) &= \chi(S) \times \chi(T). \end{aligned}$$



Stephen Schanuel:
Euler characteristic is the topological analogue of cardinality.

The idea

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Challenge Find a general definition of ‘size’, including these and other examples.

One answer The magnitude of an enriched category.

The magnitude of a matrix

Let Z be a matrix.

If Z is invertible, the magnitude of Z is

$$|Z| = \sum_{i,j} (Z^{-1})_{ij}$$

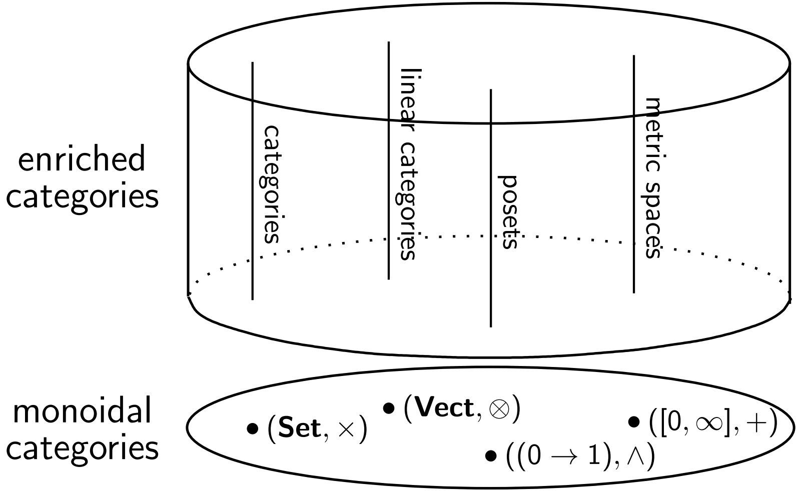
—the sum of all the entries of Z^{-1}.

(The definition can be extended to many non-invertible matrices...
but we won't need this refinement today.)

Enriched categories

A monoidal category is a category $\mathbf{V}$ equipped with some kind of product.

A category enriched in $\mathbf{V}$ is like an ordinary category, with a set/class of objects, but the 'hom-sets' $\mathrm{Hom}(A, B)$ are now objects of $\mathbf{V}$.



The magnitude of an enriched category

Let $\mathbf{V}$ be a monoidal category.

Suppose we have a notion of the 'size' of each object of $\mathbf{V}$:
a multiplicative function $|\cdot|$ from $\operatorname{ob}\mathbf{V}$ to some field k.

E.g. $\mathbf{V} = \mathbf{FinSet}$, $k = \mathbb{Q}$, $|\cdot| =$ cardinality;
$\mathbf{V} = \mathbf{FDVect}$, $k = \mathbb{Q}$, $|\cdot| =$ dimension.

Then we get a notion of the 'size' of a category $\mathbf{A}$ enriched in $\mathbf{V}$:

- write $Z_{\mathbf{A}}$ for the matrix $\big(|\mathrm{Hom}(A, B)|\big)_{A,B \in \operatorname{ob}\mathbf{A}}$ over k
- define the magnitude of the enriched category $\mathbf{A}$ to be

$$|\mathbf{A}| = |Z_{\mathbf{A}}| \in k$$

—i.e. the magnitude of the matrix $Z_{\mathbf{A}}$.

(Here assume $\mathbf{A}$ has only finitely many objects and $Z_{\mathbf{A}}$ is invertible.)

Examples not involving metric spaces

Ordinary finite categories (i.e. $\mathbf{V} = \mathbf{FinSet}$):

- For a finite category $\mathbf{A}$ satisfying mild conditions, $|\mathbf{A}|$ is $\chi(B\mathbf{A}) \in \mathbb{Z}$, the Euler characteristic of the classifying space of $\mathbf{A}$.
- For a finite group G seen as a one-object category, $|G| = 1/\mathrm{order}(G)$.
- For a finitely triangulated manifold X, its poset $\mathbf{A}$ of simplices has magnitude $|\mathbf{A}| = \chi(X) \in \mathbb{Z}$.
- For a finitely triangulated *orbifold* X, its *category* $\mathbf{A}$ of simplices has magnitude $|\mathbf{A}| = \chi(X) \in \mathbb{Q}$. (Joint with Ieke Moerdijk.)

Linear categories (i.e. $\mathbf{V} = \mathbf{Vect}$):

- For a suitably finite associative algebra E, let $\mathbf{IP}(E)$ denote the linear category of indecomposable projective E-modules.


Theorem (with Joe Chuang and Alastair King)

The magnitude of $\mathbf{IP}(E)$ *is a certain Euler form associated with* E*.*

Metric spaces as enriched categories

There's at least an *analogy* between categories and metric spaces:

A category has:	A metric space has:
objects $a, b, \ldots$	points $a, b, \ldots$
sets $\mathrm{Hom}(a, b)$	numbers $d(a, b)$
composition operation	triangle inequality
$\quad \mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$	$\quad d(a, b) + d(b, c) \geq d(a, c)$

In fact, both are special cases of the concept of enriched category.

(A metric space is a category enriched in the poset $([0, \infty], \geq)$ with $\otimes = +$.)

2. The magnitude of a metric space

The magnitude of a finite metric space (concretely)

Starting from the general definition of the magnitude of an enriched category, we can specialize to metric spaces. Here's what we get.

To compute the magnitude of a finite metric space $A = \{a_1, \ldots, a_n\}$:

- write down the $n \times n$ matrix with (i, j)-entry $e^{-d(a_i, a_j)}$
- invert it
- add up all n^2 entries.

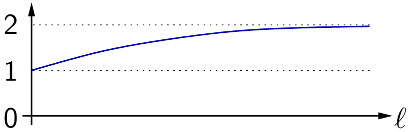
And that's the magnitude $|A|$.

(Where does the "e^{-x}" come from?

It's because $f : x \mapsto e^{-x}$ is essentially the only function satisfying $f(x + y) = f(x) \cdot f(y)$.)

The magnitude of a finite metric space: first examples

- $|\varnothing| = 0$.
- $|\bullet| = 1$.
- $\left|\bullet \overset{\leftarrow\ \ell\ \rightarrow}{\qquad} \bullet\right| =$ sum of entries of $\begin{pmatrix} e^{-0} & e^{-\ell} \\ e^{-\ell} & e^{-0} \end{pmatrix}^{-1} = \dfrac{2}{1+e^{-\ell}}$



- If $d(a,b) = \infty$ for all $a \neq b$ then $|A| = \text{cardinality}(A)$.

Slogan: Magnitude is the 'effective number of points'

Magnitude functions

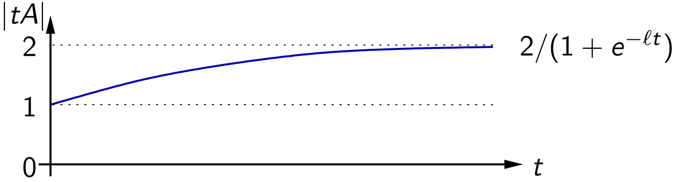
Magnitude assigns to each metric space not just a *number*, but a *function*.

For $t > 0$, write tA for A scaled up by a factor of t.

The magnitude function of a metric space A is the partial function

$$\begin{array}{ccc} (0,\infty) & \to & \mathbb{R} \\ t & \mapsto & |tA|. \end{array}$$

E.g.: the magnitude function of $A = (\bullet \overset{\leftarrow\ \ell\ \rightarrow}{\quad} \bullet)$ is



A magnitude function has only finitely many singularities (none if $A \subseteq \mathbb{R}^n$).

It is increasing for $t \gg 0$, and $\lim_{t\to\infty} |tA| = \text{cardinality}(A)$.

The magnitude of a compact metric space

In principle, magnitude is only defined for enriched categories *with finitely many objects* — here, *finite* metric spaces.

Can the definition be extended to, say, compact metric spaces?

Theorem (Mark Meckes)

All sensible ways of extending the definition of magnitude from finite metric spaces to compact 'positive definite' spaces are equivalent.

Proof Uses some functional analysis.

Positive definite spaces include all subspaces of $\mathbb{R}^n$ with Euclidean or ℓ^1 (taxicab) metric, and many other common spaces.

The magnitude of a compact positive definite space A is

$$|A| = \sup\{|B| \ : \ \text{finite } B \subseteq A\}.$$

First examples

E.g. Line segment: $|t[0,\ell]| = 1 + \frac{1}{2}\ell \cdot t$.

E.g. Let $A \subseteq \mathbb{R}^2$ be an axis-parallel rectangle with the ℓ^1 (taxicab) metric. Then

$$|tA| = \chi(A) + \tfrac{1}{4}\mathrm{perimeter}(A) \cdot t + \tfrac{1}{4}\mathrm{area}(A) \cdot t^2.$$

0-dimensional measure

1-dimensional measure

dimension of A

2-dimensional measure

Magnitude encodes geometric information

Theorem (Meckes) *Let A be a compact subset of $\mathbb{R}^n$, with Euclidean metric. From the magnitude function of A, you can recover its Minkowski dimension.*

Proof Uses a deep theorem from potential analysis, plus the notion of maximum diversity.

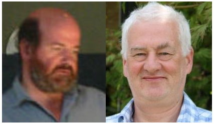
Theorem (Willerton) *Let A be a homogeneous Riemannian n-manifold. Then as $t \to \infty$,*

$$|tA| = a_n \mathrm{vol}(A) \cdot t^n + b_n \mathrm{tsc}(A) \cdot t^{n-2} + O(t^{n-4}),$$

where a_n and b_n are constants and tsc *is total scalar curvature.*

Proof Uses some asymptotic analysis.

Magnitude encodes geometric information



Theorem (Barceló and Carbery) *From the magnitude function of $A \subseteq \mathbb{R}^n$, you can recover the volume of A.*

Proof Uses PDEs and Fourier analysis.

Theorem (Barceló and Carbery) *For odd n, the magnitude function of the Euclidean ball B^n is a rational function over $\mathbb{Q}$.*

Examples

$$\left|tB^1\right| = 1 + t$$

$$\left|tB^3\right| = 1 + 2t + t^2 + \tfrac{1}{6}t^3$$

$$\left|tB^5\right| = \frac{360 + 1080t + 1080t^2 + 525t^3 + 135t^4 + 18t^5 + t^6}{120(3+t)}$$

Magnitude encodes geometric information

Theorem (Gimperlein, Goffeng and Louca) *Let A be a sufficiently regular subset of $\mathbb{R}^n$. From the magnitude function of A, you can recover the surface area of A.*

Proof Uses heat trace asymptotics (techniques related to heat equation proof of Atiyah–Singer index theorem) and treats t as a *complex* parameter.

Theorem (Gimperlein and Goffeng) *Let A and B be nice subsets of $\mathbb{R}^n$. Then*

$$|t(A \cup B)| + |t(A \cap B)| - |tA| - |tB| \to 0$$

as $t \to \infty$.

Magnitude of metric spaces doesn't *literally* obey inclusion-exclusion, as that would make it trivial. But it *asymptotically* does.

$2\frac{1}{2}$. The magnitude of a graph

The magnitude of a graph

Graph will mean finite, undirected graph with no multiple edges or loops.

Let G be a graph, with vertices $a_1, \ldots, a_n$.

The distance between two vertices is the shortest path-length between them.

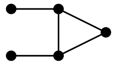
We will use a formal variable q. (Think of it as e^{-t}.)

Let Z_G be the $n \times n$ matrix with (i, j)-entry $q^{d(a_i, a_j)}$.

Then Z_G is invertible over the field $\mathbb{Q}(q)$ of rational functions in q.

The magnitude of G is $|G| = |Z_G| \in \mathbb{Q}(q)$.

E.g.:

$\Bigg|$  $\Bigg|$ $= \dfrac{5 + 5q - 4q^2}{(1+q)(1+2q)} = 5 - 10q + 16q^2 - 28q^3 + \cdots .$

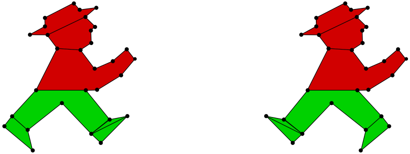
The magnitude of a graph: properties

Cardinality-like properties

- $|\bullet \quad \bullet \quad \bullet| = 3$, etc.
- $|G \,\square\, H| = |G| \cdot |H|$, where $\square$ is the 'cartesian product' of graphs
- $|G \cup H| = |G| + |H| - |G \cap H|$, under hypotheses.

Magnitude also bears some resemblance to the Tutte polynomial.

For instance, these two graphs have the same magnitude:



This is a Whitney twist. Their invariance under magnitude has been studied, clarified and generalized by Emily Roff.

But neither magnitude nor the Tutte polynomial is determined by the other.

3. Magnitude homology

The idea in brief

Find a homology theory for enriched categories that categorifies magnitude.

This was first done for graphs (seen as metric spaces) by Hepworth and Willerton in 2015: given a graph G,

- they defined a group $H_{n,\ell}(G)$ for all integers $n, \ell \geq 0$ (a *graded* homology theory);
- writing $\chi_\ell(G) = \sum_n (-1)^n \operatorname{rank}(H_{n,\ell}(G))$, they showed that the magnitude function of G is equal to

$$t \mapsto \sum_\ell \chi_\ell(G) e^{-\ell t}.$$

So: *the Euler characteristic of magnitude homology is magnitude.*

From graphs to enriched categories and metric spaces

The definition of magnitude homology was extended from graphs to enriched categories in work with Mike Shulman in 2017.

Definition omitted. . .

Since metric spaces are enriched categories, we get a *homology theory for metric spaces.*

In fact, it is a *$[0, \infty)$-graded homology theory*: for each metric space A, integer $n \geq 0$ and real $\ell \in [0, \infty)$, there is an abelian group $H_{n,\ell}(A)$.

For *finite* metric spaces, magnitude homology categorifies magnitude:

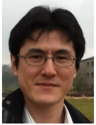
$$|tA| = \sum_{\ell \in [0,\infty)} \chi_\ell(A) e^{-\ell t}$$

(interpreted suitably), where $\chi_\ell(A) = \sum_n (-1)^n \operatorname{rank}(H_{n,\ell}(A))$ as before.

But there is no such theorem for non-finite spaces!

This is the major challenge in the field.

Magnitude homology of metric spaces

-  While ordinary homology detects the *existence* of holes, magnitude homology detects the *diameter* of holes (Ryuki Kaneta and Masahiko Yoshinaga).

- Magnitude homology can distinguish between graphs that have the same magnitude (Yuzhou Gu). 

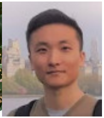
-  The magnitude homology of a convex subset of $\mathbb{R}^n$ is trivial (Kaneta and Yoshinaga; Benoît Jubin).

- If a metric space contains a closed geodesic then its 2nd magnitude homology group is nontrivial (Yasuhiko Asao). 

- A slogan of Kiyonori Gomi:

$$\boxed{\text{The more geodesics are unique, the more magnitude homology is trivial}}$$

What's happening in magnitude homology?

-   There is a relationship between magnitude homology and *persistent homology*—but they detect different information (Nina Otter; Simon Cho).
- Applications of magnitude homology to the analysis of *networks* (Giuliamaria Menara) 
- A theory of magnitude *cohomology* (Hepworth).
- Connections between magnitude homology and *path homology* (Asao).
- A comprehensive *spectral sequence* approach that encompasses both magnitude homology and path homology (Asao; Hepworth and Roff; Gomi).
- A concept of *magnitude homotopy* (Yu Tajima and Yoshinaga).
- New results on *magnitude homology equivalence* of subsets of $\mathbb{R}^n$, involving convex geometry (with Adrián Doña Mateo) 
- And lots more... find out this week!

4. (Bio)diversity

What is diversity?

Conceptual question Given an ecological community, consisting of individuals grouped into species, how can we reasonably quantify its 'diversity'?

Simplest answer Count the number n of species present.

Mathematically: cardinality of a finite set.

Better answer Use the relative abundance distribution $\mathbf{p} = (p_1, \ldots, p_n)$ of species. ("Relative" means that $\sum p_i = 1$.)

For any choice of parameter $q \in \mathbb{R}^+$, we can quantify diversity as

$$D_q(\mathbf{p}) = \left(\sum_i p_i^q \right)^{1/(1-q)}.$$

E.g. If $\mathbf{p} = (\frac{1}{n}, \ldots, \frac{1}{n})$ then $D_q(\mathbf{p}) = n$.

Mathematically: $\approx$ entropy of a probability distribution on a finite set.

Similarity between species

Even better answer Also use a matrix Z of similarities between species.

The idea: Z_{ij} is the similarity between species i and species j.

Interpretation:

- If $Z_{ij} = 0$, species i and j are completely dissimilar—nothing in common.
- If $Z_{ij} = 1$, species i and j are identical. (So normally $Z_{ii} = 1$.)

This gives an $n \times n$ similarity matrix $Z = (Z_{ij})_{i,j=1}^{n}$.

How do we measure similarity?

However we like! Examples:

- The "naive model", where different species have nothing at all in common: Z is the identity matrix I.
- Genetically, phylogenetically, functionally, morphologically, ...
- If we have a metric d on the set $\{1, \ldots, n\}$ of species, we can define the similarities by $Z_{ij} = e^{-d(i,j)}$.

Species-sensitive diversity measures

Take an ecological community with similarity matrix Z and a relative abundance distribution $\mathbf{p}$.

For any choice of parameter $q \in \mathbb{R}^+$, we can quantify diversity as

$$D_q^Z(\mathbf{p}) = \left(\sum_i p_i (Z\mathbf{p})_i^{q-1} \right)^{1/(1-q)}.$$

The formula is not important here. But...

Discovery (with Christina Cobbold) Most of the biodiversity measures most commonly used in ecology are special cases of D_q^Z.

Mathematically: ≈ entropy of a probability distribution on a finite metric space.

The maximization problem

Fix a list of species, with known similarity matrix Z.

What is the maximum diversity that can be achieved by varying the species abundances? In other words: what is $\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$?

In principle, the answer depends on the parameter q. But...

Theorem (with Meckes) *The answer is independent of q.*

So, $\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$ is a well-defined number associated with the matrix Z — the maximum diversity $D_{\max}(Z)$ of Z.

Fact $D_{\max}(Z)$ is the magnitude of some submatrix of Z.

Conclusion: Magnitude is closely related to maximum diversity.

Back to geometry

The diversity measures D_q^Z and the maximum diversity theorem can be extended from finite to compact spaces (joint theorem with Roff).

So, every compact metric space A has a well-defined maximum diversity $D_{\max}(A) \in \mathbb{R}^+$.

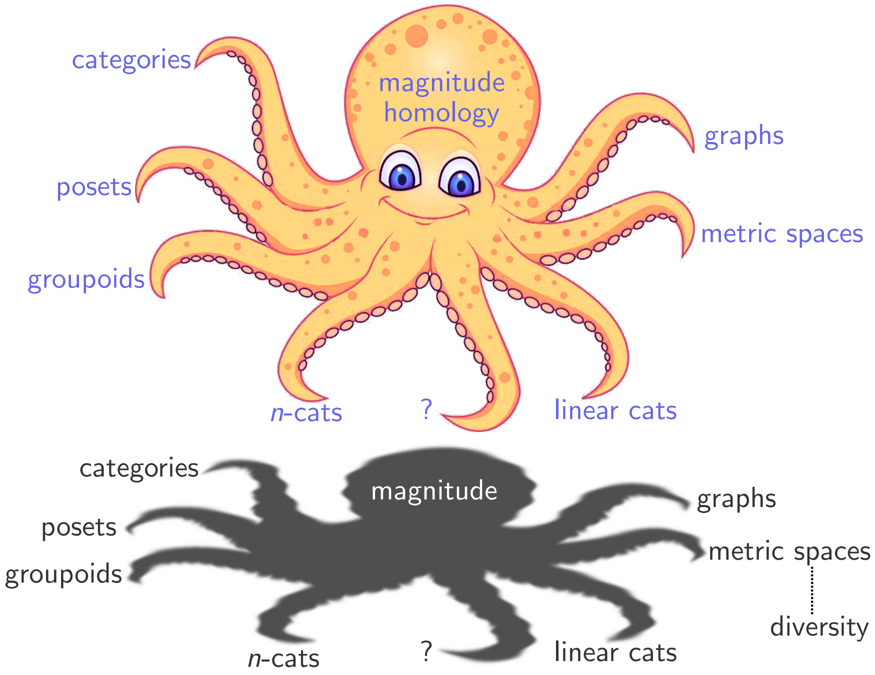
It is equal to the magnitude of some closed subset of A.

We know some things about the behaviour of maximum diversity.

For example, Meckes showed that $D_{\max}(tA)$ grows like $t^{\dim A}$ as $t \to \infty$, where $\dim A$ is Minkowski dimension.

But we don't know the maximum diversity of even some very simple spaces, such as the 2-dimensional Euclidean disc!

Summary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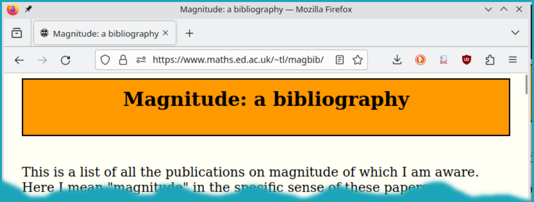
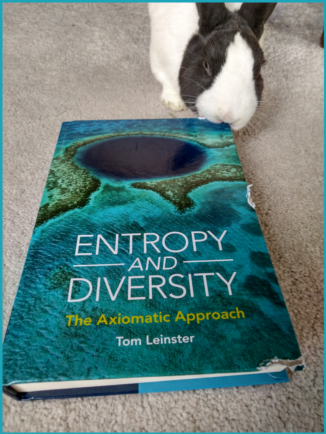
References



The magnitude bibliography:

www.maths.ed.ac.uk/∼tl/magbib

Lists all publications on magnitude.



The diversity story