JÓNSSON-TARSKI TOPOSES

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Terminology

Let $\mathbb{A}$ and $\mathbb{B}$ be small categories.

$\hat{\mathbb{A}} = [\mathbb{A}^{op}, \mathbf{Set}]$ (= "right $\mathbb{A}$-modules").

An $(\mathbb{A}, \mathbb{B})$-module M is a functor $\mathbb{B}^{op} \times \mathbb{A} \xrightarrow{M} \mathbf{Set}$.

Then M induces an adjunction

$$\begin{tikzcd} \hat{\mathbb{A}} \arrow[r, shift left, "- \otimes M"] \arrow[r, phantom, "\perp"] & \hat{\mathbb{B}} \arrow[l, shift left, "{[M, -]}"] \end{tikzcd}$$

where for $Y \in \hat{\mathbb{B}}$ and $a \in \mathbb{A}$,

$$[M, Y](a) = \mathrm{Hom}(M(-, a), Y).$$

A two-sided $\mathbb{A}$-module is an $(\mathbb{A}, \mathbb{A})$-module.

Big Picture

Any two-sided module
gives rise to
an object of topology

... in at least two ways.

① Self similarity   To a first approximation:

M gives rise to

a functor $I_M \; \mathbb{A} \longrightarrow \mathrm{Top}$

the terminal coalgebra for $[\mathbb{A}, \mathrm{Top}] \circlearrowleft M \otimes -$.

② This talk   M gives rise to

a topos $\mathcal{JT}_M$,

the "Jónsson-Tarski topos of M". (Uses $\hat{\mathbb{A}} \circlearrowleft [M,-]$.)

The classical Jonsson Tarski topos
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A Jónsson-Tarski algebra (X, ξ) is a set X with a bijection

$$\xi : X \xrightarrow{\sim} X \times X.$$

They form a category JT_2.

Three surprising things:

1. Jónsson-Tarski algebras are an algebraic theory
2. (free algebra on 1) $\cong$ (free algebra on 2)
3. JT_2 is a topos.

Proofs:

1. A Jónsson-Tarski algebra is a set X with operations
$$l, r : X \longrightarrow X, \qquad \bullet : X^2 \longrightarrow X$$
satisfying certain equations.

2.
$$\begin{tikzcd} & JT_2 \arrow[dl, "u"'] \arrow[dr, "u"] & \\ Set \arrow[rr, "(\,)^2"'] & & Set \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & JT_2 & \\ Set \arrow[ur, "F"] & & Set \arrow[ul, "F"'] \arrow[ll, "-\times 2"] \end{tikzcd}$$

$$\begin{tikzcd} JT_2 \arrow[d, shift right=2, "u"] \\ Set \arrow[u, shift right=2, "F"'] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

3. (Freyd) Site is free monoid on 2 generators λ, ρ;
$\{\lambda, \rho\}$ is a cover.

General Jónsson Tarski toposes

Let $\mathbb{A}$ be a small category and M a two-sided $\mathbb{A}$-module.

A Jónsson-Tarski M-algebra is a pair (X, ξ) where $X \in \hat{\mathbb{A}}$ and

$$\xi : X \xrightarrow{\sim} [M, X].$$

They form a category JT_M.

E.g.: $\mathbb{A} = \mathbb{1}$, $M = 2$: then $JT_M = JT_2$.

No-longer-surprising things:

1. JT_M is monadic over $\hat{\mathbb{A}}$
2. For $X \in \hat{\mathbb{A}}$,

   (free JT M-algebra on X)

   $\cong$ (free JT M-algebra on $X \otimes M$)
3. JT_M is a topos.

Proof of 3: define a site $\mathbb{A}_M$ by adjoining to $\mathbb{A}$ one new arrow $b \to a$ for each $b, a \in \mathbb{A}$ and $m \in M(b,a)$. For each a, the family of such arrows covers a. Then $JT_M = Sh(\mathbb{A}_M)$.

Finite discrete case

Take $\mathbb{A} = \{1, \ldots, n\}$, discrete cat, and $M : \mathbb{A}^{op} \times \mathbb{A} \to \mathrm{FinSet}$.

Then M is an $n \times n$ matrix (μ_{ij}) of natural numbers, or equivalently a finite directed graph with n vertices.

A Jónsson-Tarski M-algebra consists of sets $X_1, \ldots, X_n$ with bijections

$$\xi_1 : X_1 \xrightarrow{\sim} X_1^{\mu_{11}} \times \cdots \times X_n^{\mu_{1n}}$$

$$\vdots$$

$$\xi_n : X_n \xrightarrow{\sim} X_1^{\mu_{n1}} \times \cdots \times X_n^{\mu_{nn}}.$$

The site $\mathbb{A}_M$ is the free category on the directed graph that "is" M.

Some non-discrete examples

- "Real interval"

Let $\mathbb{A} = (0 \rightrightarrows 1)$; then $\hat{\mathbb{A}} = \mathrm{DirGph}$.

There is an $(\mathbb{A},\mathbb{A})$-module M such that if $X = (X_1 \rightrightarrows X_0) \in \hat{\mathbb{A}}$ then

$$[M, X] = (X_1 \times_{X_0} X_1 \rightrightarrows X_0).$$

A Jónsson-Tarski M-algebra is a graph X with an iso

$$\xi : \begin{pmatrix} X_1 \\ \downdownarrows \\ X_0 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} X_1 \times_{X_0} X_1 \\ \downdownarrows \\ X_0 \end{pmatrix}.$$

(When $\xi_0 = 1_{X_0}$, this is a "bijective composition" on X.)

Have lifting

$$\begin{tikzcd} & JT_M \arrow[d, "u"] \\ \mathrm{Top} \arrow[ur, dashed] \arrow[r, "\pi_1"'] & \hat{\mathbb{A}} = \mathrm{DirGph} \end{tikzcd}$$

- "Singular complex"

Let $\Delta_{\text{face}} = (\text{order-preserving injections}) \subset \Delta$;
then $\widehat{\Delta_{\text{face}}} = (\text{semi-simplicial sets})$.

There is an $(\mathbb{A},\mathbb{A})$-module M such that

$$[M, X] = (\text{barycentric subdivision of } X).$$

Have lifting

$$\begin{tikzcd} & JT_M \arrow[d, "u"] \\ \mathrm{Top} \arrow[ur, dashed] \arrow[r, "\text{singular}"'] & \widehat{\Delta_{\text{face}}} = \mathrm{ssSet}. \end{tikzcd}$$

Open questions

- Which toposes are Jónsson-Tarski?

  Thm: Every presheaf topos is Jónsson-Tarski.

  Thm: $Sh(X)$ is Jónsson-Tarski for every compact metric totally disconnected X.

  Thm: (Lack) If $\mathcal{E}$ is Jónsson-Tarski and $E \in \mathcal{E}$ then $\mathcal{E}/E$ is Jónsson-Tarski.

- Is JT_M an étendue relative to $\hat{\mathbb{A}}$?

- A two-sided module M gives rise to <u>two</u> topological objects, the topos JT_M and the functor $I_M : \mathbb{A} \to Top$ ($\approx$ terminal coalgebra for $[\mathbb{A}, Top] \circlearrowleft M \otimes -$).

  How are they related?

  E.g.: $JT_2 / F(1) \simeq Sh(2^{\mathbb{N}})$.