Category Theory 8
Limits and colimits

This is to accompany the reading of 21–28 November. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Let $\mathbb{I}$ be a small category and $D : \mathbb{I} \longrightarrow \mathbf{Set}$ a diagram of shape $\mathbb{I}$ in **Set**. Describe explicitly a limit cone and a colimit cocone for D.

2. What does it mean for a functor to **preserve**, **reflect** or **strictly create** limits? Show that if $F : \mathcal{A} \longrightarrow \mathcal{B}$ strictly creates limits and $\mathcal{B}$ has all limits, then $\mathcal{A}$ has all limits and F preserves them.

3. Let $\mathcal{A}$ be a category with binary products. Show that
$$\mathcal{A}(A, B \times C) \cong \mathcal{A}(A, B) \times \mathcal{A}(A, C)$$
naturally in $A, B, C \in \mathcal{A}$.
(I'm assuming implicitly that we've chosen for each B and C a product cone on (B, C). By Sheet 7, q.2, the assignment $(B, C) \longmapsto B \times C$ is then functorial—which it would have to be in order for the word 'naturally' in the question to make sense.)

4. Let $\mathbb{I}$ be a small category. Show that a category $\mathcal{A}$ has all limits of shape $\mathbb{I}$ if and only if the diagonal functor $\Delta : \mathcal{A} \longrightarrow [\mathbb{I}, \mathcal{A}]$ has a right adjoint.

5. Recall the definitions of regular monic and split monic from Sheet 7, q.5.
   (a) Give an example, with proof, of a map in a category that is monic and epic but not an isomorphism.
   (b) Prove that in any category, a map is an isomorphism if and only if it is both monic and regular epic.
   (c) Assuming that our category of sets satisfies the Axiom of Choice (page 40 of the notes), show that
   $$\text{epic} \iff \text{regular epic} \iff \text{split epic}$$
   in **Set**.

   (You can say that a category $\mathcal{A}$ 'satisfies the Axiom of Choice' if all epics in $\mathcal{A}$ are split. For example, the Axiom of Choice is not *satisfied in* **Top** *or in* **Gp**.*)*