# Category Theory

## Hints on the problem sheets

I've written varying amounts about each question. Sometimes it's just a quick hint and sometimes it's something more detailed—but almost none of my answers are up to the level of detail expected in an exam.

**General hint** Before you look here for a hint,

*make sure you understand the question in full.*

In category theory, maybe more than in most subjects, you really have to completely understand every piece of terminology used in the question before trying to answer it. If you don't, you're extremely unlikely to produce a correct answer. But once you do, you may well find the answer a pushover. The purpose of these questions is to deepen and test your understanding, not to exercise your problem-solving skills. It's not like number theory or combinatorics, where there are many questions that can be stated in simple terms but are very hard to answer.

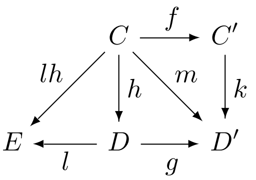
So, the questions are often harder than the answers! This is particularly true of the questions on the earlier sheets.

### Sheet 1: Categories and functors

1. For everyday examples of categories and functors, browse library or web. Or you can make up examples in the following manner. There's a category

$$\mathcal{A} = (A \xrightarrow{p} B)$$

—that is, $\mathcal{A}$ has two objects, $A$ and $B$, and just one non-identity map, $p : A \longrightarrow B$. (No need to say what composition is, as that's uniquely determined.) Or (random example) there's a category $\mathcal{B}$ with objects and maps



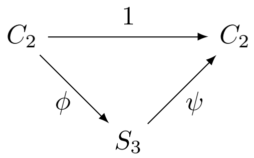

where $gh = kf = m$ and I've omitted identity maps. There's a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ defined by $F(A) = C$, $F(B) = C'$, and $F(p) = f$.

2. See 'General hint' above.

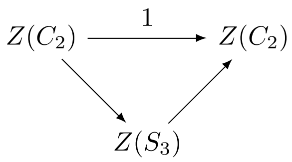
3. (a) Same set but multiplication reversed: $(a, b) \longmapsto b \cdot a$. Isomorphism $G \longrightarrow G^{\mathrm{op}}$ provided by $g \longmapsto g^{-1}$.

   (b) Let $M$ be the monoid of maps $2 \longrightarrow 2$ where $2$ is a two-element set and multiplication is composition. Then the statement $\exists m \in M : \forall x \in M, mx = m$ is true, but becomes false when $M$ is replaced by $M^{\mathrm{op}}$. So $M \not\cong M^{\mathrm{op}}$.

4. No. Main point: a homomorphism $\phi : G \longrightarrow H$ doesn't restrict to a map $Z(G) \longrightarrow Z(H)$ (e.g. take an injection $\phi : C_2 \hookrightarrow S_3$). So the *obvious* way of defining $Z$ on maps fails. In fact there's *no* way to do it: for if there were, the commutative diagram




(where $\psi$ is the quotient map for $A_3 \trianglelefteq S_3$) would become a commutative diagram

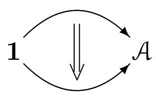



which is impossible as $Z(C_2) = C_2$ and $Z(S_3) \cong 1$.

5. Easy once you fully understand the question. Write out the definition of $\mathcal{A} \times \mathcal{B}$ *in full*: what the objects, maps, composition and identities are. Write down *in full* what a functor $\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ is. Then try it.

**Sheet 2: Natural transformations and equivalence**

1. For examples that occur mathematical practice, browse library or web. Can also make up examples as in hints to Sheet 1, q.1. E.g. if $\mathbf{1}$ is the category with one object and one map (the identity) then a functor from $\mathbf{1}$ to a category $\mathcal{A}$ is just an object of $\mathcal{A}$, and a natural transformation




between two such functors is a map in $\mathcal{A}$ between the corresponding two objects. Or, take the categories $\mathcal{A}$ and $\mathcal{B}$ defined in the hints to Sheet 1, q.1: then there is a functor $F$ as defined there, another functor $G$ defined by $G(p) = g$, and a natural transformation $\alpha : F \longrightarrow G$ given by $\alpha_A = h$ and $\alpha_B = k$.

2. See 'General hint' above.

3. Define $F : \mathbf{Mat} \longrightarrow \mathbf{FDVect}$ as follows: $F(n) = k^n$, and if $M \in \mathbf{Mat}(m,n)$ then $F(M)$ is the linear map $k^m \longrightarrow k^n$ corresponding to the matrix $M$ (with respect to the standard bases). Show functorial. Show full and faithful and essentially surjective on objects. Invoke 1.3.12.

   This functor $F$ is canonical, but there's no canonical functor $G : \mathbf{FDVect} \longrightarrow \mathbf{Mat}$ satisfying $FG \cong 1$ and $GF \cong 1$: for such a $G$ must send every finite-dimensional vector space $V$ to $\dim V$ (fine), but to specify $G$ on maps, you'd have to choose a basis for every finite-dimensional vector space, which can't be done in a canonical way.

4. Conjugacy.

5.(a) Let $f : X \longrightarrow Y$ be a map in $\mathcal{B}$. Then $\mathbf{Sym}(f) : \mathbf{Sym}(X) \longrightarrow \mathbf{Sym}(Y)$ is defined by $\sigma \longmapsto f\sigma f^{-1}$. Also $\mathbf{Ord}(f) : \mathbf{Ord}(X) \longrightarrow \mathbf{Ord}(Y)$ is defined by $\leq \longmapsto \leq'$ where $y_1 \leq' y_2 \iff f^{-1}(y_1) \leq f^{-1}(y_2)$. Check functoriality.

(b) Take $\alpha : \mathbf{Sym} \longrightarrow \mathbf{Ord}$. Draw naturality square for $\alpha$ with respect to the map $f : 2 \longrightarrow 2$ in $\mathcal{B}$ where 2 is a two-element set and $f$ interchanges its elements. Work out what its commutativity says when you take the identity permutation $1 \in \mathbf{Sym}(2)$: get contradiction.

## Sheet 3: Adjoints

1. Same comments as for Sheet 1, q.1 and Sheet 2, q.1.

2. They are just bijections between sets (or strictly speaking, classes): if $F \dashv G$ is an adjunction between discrete categories $\mathcal{A}$ and $\mathcal{B}$ then $F$ is an isomorphism and $G = F^{-1}$.

3. For all $B$, the set $\mathcal{B}(F(I), B) \cong \mathcal{A}(I, G(B))$ has one element. And dually.

4. Bookwork.

5. The substantial parts are (i) understanding the concepts behind the question, and (ii) observing that if $\eta_A$ is an isomorphism then so is $\varepsilon_{F(A)}$ (by a triangle identity) and dually.

   The equivalence you restrict to can be completely trivial, e.g. the adjunction $\mathbf{Vect}_k \rightleftarrows \mathbf{Set}$ becomes the equivalence $\emptyset \rightleftarrows \emptyset$ (where $\emptyset$ is the empty category). Slightly less trivial: $\mathbf{Top} \underset{D}{\overset{U}{\rightleftarrows}} \mathbf{Set}$ gives the equivalence (discrete spaces) $\simeq \mathbf{Set}$.

## Sheet 4: Adjoints and sets

1. Bookwork.

2. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. Then $F$ has a right adjoint if and only if for each $B \in \mathcal{B}$, the category $(F \Rightarrow B)$ has a terminal object.

   Proof: can just say 'by duality'.

3. Left: $(A, B) \longmapsto A + B$. Right: $(A, B) \longmapsto A \times B$.

4. I can think of three general strategies for finding adjoints. You can use them to find $D$, $I$ and $C$ respectively.

   **Guess it** We're given $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ and want to know what its adjoints are. Have a guess: what functors $\mathbf{Set} \longrightarrow \mathbf{Cat}$ do we already know? In other words, what methods do we know for constructing a category out of a set? One is the discrete category construction (1.3.3(a)), which defines a functor $D : \mathbf{Set} \longrightarrow \mathbf{Cat}$. Check that this is left adjoint to $O$.

**Probe it** We're told that $O$ has a right adjoint $I$. We can try to figure out what it must be by using adjointness. Given a set $S$, an object of $I(S)$ is a functor $\mathbf{1} \longrightarrow I(S)$, which is a function $O(\mathbf{1}) \longrightarrow S$, which is an element of $S$. So the object-set of $I(S)$ is $S$. An arrow in $I(S)$ is a functor $\mathbf{2} \longrightarrow I(S)$ (where $\mathbf{2}$ is the category $\mathcal{A}$ in the hint to Sheet 1, q.1), which is a function $O(\mathbf{2}) \longrightarrow S$, which is a pair of elements of $S$. So the arrow-set of $I(S)$ is $S \times S$. You could carry on with this method to figure out what domain, codomain, composition and identities are in $I(S)$, but perhaps you can now make the leap and guess it: $I(S)$ is the category whose objects are the elements of $S$, where for each $A, B \in S$ there is exactly one map $A \longrightarrow B$, and where composition and identities are defined in the only possible way. It's called the **indiscrete category** on $S$.

**Stare at it** We'll use this to find $C$. Let $\mathbb{A}$ be a category and $S$ a set. A functor $F : \mathbb{A} \longrightarrow D(S)$ is supposed to be the same thing as a function $C(\mathbb{A}) \longrightarrow S$, whatever $C$ is. Well, what *is* a functor $F : \mathbb{A} \longrightarrow D(S)$? It's a way of assigning to every object $A \in \mathbb{A}$ an element $F(A)$ of $S$, with the property that for every map $A \xrightarrow{f} B$ in $\mathbb{A}$ we have $F(A) = F(B)$. In other words (aha!), it's a function $O(\mathbb{A})/\!\sim \longrightarrow S$ where $\sim$ is the equivalence relation on $O(\mathbb{A})$ generated by $A \sim B$ whenever there's a map $A \longrightarrow B$. So $C(\mathbb{A}) = O(\mathbb{A})/\!\sim$. This is called the set of **connected-components** of $\mathbb{A}$.

## Sheet 5: Representables

1. Bookwork.

2. The non-inventive answer: by definition, theere's one representable for every pair $(\mathcal{A}, A)$ where $\mathcal{A}$ is a category and $A \in \mathcal{A}$, namely $H^A$. So to give five examples of representable functors, you can just write down five examples of objects of categories!

   For more interesting answers, browse library/web.

3. Take isomorphism $\alpha : H_A \longrightarrow H_B$. We have to define maps $A \underset{g}{\overset{f}{\rightleftarrows}} B$ and prove $gf = 1_A$ and $fg = 1_B$. Define $f = \alpha_A(1_A)$ and $g = \alpha_B(1_B)$. (What else could we possibly do?) Get $gf = 1_A$ and $fg = 1_B$ from naturality of $\alpha$.

4. (a) Pushover once you fully understand the question: e.g. make sure you fully understand how monoids are one-object categories and $M$-sets are functors $M \longrightarrow \mathbf{Set}$. If it helps, use a different letter ($\mathcal{M}$, say) for the one-object category corresponding to the monoid $M$.

   (b) The unique map $\alpha$ is $m \longmapsto xm$. The bijection is $\phi \longmapsto \phi(1)$. (Moral: unique existence statements can be rephrased as saying that some function is a bijection.)

   (c) This is just (b) rephrased.

   (Well, the statement of the Yoneda Lemma also includes naturality in $X$ and in the object (usually called '$A$'). We haven't proved this part, although we know that our bijection is natural in the sense of being canonically defined—no random choices involved.)

5. Same kind of comments as for Sheet 1, q.5.

## Sheet 6: The Yoneda Lemma

1. Bookwork.

2. Definition of Yoneda embedding: bookwork.

   (a) If $f : A \longrightarrow B$ is a map in $\mathcal{A}$ then $f = H_f(1_A)$.

   (b) Given $\alpha : H_A \longrightarrow H_B$, define $f = \alpha_A(1_A)$; show $H_f = \alpha$.

   (c) Definition of universality: see 3.3.2. Isomorphism $\alpha : H_A \longrightarrow X$ given by $\alpha_B(g) = (Xg)(x)$.

3.(a) If $J(f)$ is an isomorphism then by fullness, there exists a map $f' : C' \longrightarrow C$ such that $J(f') = J(f)^{-1}$; then check that $J(f'f) = 1$, which by faithfulness implies $f'f = 1$, and similarly $ff' = 1$.

   (b) Use (a).

   (c) Follows from (b).

4.(a) As the hint on the problem sheet suggests, it's easy once you understand the question. If you're having trouble, try writing out in full the definition of the functor $J \circ -$ (i.e. what it does to objects *and* to maps).

   (b) Follows from (a) and q.3.

   (c) Take $\mathcal{C} = \mathcal{A}$ and $J = H_\bullet$ in (b).

## Sheet 7: Limits

1. Definition of limit: bookwork.

   Uniqueness: there are at least three statements you might make. Let $D : \mathbb{I} \longrightarrow \mathcal{A}$ be a diagram and take limit cones $(L \xrightarrow{p_I} D(I))_{I\in\mathbb{I}}$ and $(L' \xrightarrow{p'_I} D(I))_{I\in\mathbb{I}}$.

   **Weakest** $L \cong L'$.

   **Stronger** There is a unique isomorphism $j : L \longrightarrow L'$ such that $p'_I \circ j = p_I$ for all $I$.

   **Strongest** There is a unique map $j : L \longrightarrow L'$ such that $p'_I \circ j = p_I$ for all $I$, and $j$ is an isomorphism.

   I'll prove the strongest. First half of statement holds because $(L' \xrightarrow{p'_I} D(I))_{I\in\mathbb{I}}$ is a limit cone. Similarly, have unique map $j' : L' \longrightarrow L$ such that $p_I \circ j' = p'_I$ for all $I$. Then $j'j : L \longrightarrow L$ satisfies $p_I \circ j'j = p_I$ for all $I$, and $1_L$ satisfies $p_I \circ 1_L = p_I$ for all $I$; but $(L \xrightarrow{p_I} D(I))_{I\in\mathbb{I}}$ is a limit cone, so $j'j = 1_L$. Similarly, $jj' = 1_L$. So $j$ is an isomorphism.

2. We define a functor $F : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ given on objects by $F(X, Y) = X \times Y$.

   Given a map $(X, Y) \xrightarrow{(f,g)} (X', Y')$ in $\mathcal{A} \times \mathcal{A}$, there is a unique map $h : X \times Y \longrightarrow X' \times Y'$ such that

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_1^{X,Y}} & X \\ h\big\downarrow & & \big\downarrow f \\ X' \times Y' & \xrightarrow[p_1^{X',Y'}]{} & X' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X \times Y & \xrightarrow{p_2^{X,Y}} & Y \\ h\big\downarrow & & \big\downarrow g \\ X' \times Y' & \xrightarrow[p_2^{X',Y'}]{} & Y' \end{array}$$

commute, since $(X' \xleftarrow{p_1^{X',Y'}} X' \times Y' \xrightarrow{p_2^{X',Y'}} Y')$ is a product cone. Define $F(f, g) = h$. Check functoriality. To justify the word 'canonical', observe that in this answer we've done nothing random (unlike the question-setter, who randomly chose a product cone on every pair of objects).

3. Do 'if' and 'only if' separately. The only thing you've got to work with is the definition of pullback, and there's only one way to proceed.

4. No. E.g. if $f = g$ then $i$ is an isomorphism, but then the square is a pullback if and only if $f$ is monic (see 4.1.31). So we get a counterexample from any non-monic map. For instance, take $f$ and $g$ both to be the unique map $2 \longrightarrow 1$ in **Set**.

5.(a) If $m$ is split monic with $em = 1$ then $m$ is equalizer of $B \overset{me}{\underset{1}{\rightrightarrows}} B$. If $m$ is regular monic then the uniqueness part of the definition of equalizer implies that $m$ is monic.

(b) Any monic $m : A \longrightarrow B$ in **Ab** is the equalizer of $B \overset{q}{\underset{0}{\rightrightarrows}} B/\mathrm{im}(m)$ where $q$ is the quotient map (much as in 4.1.15(c)). The map $m : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $m(x) = 2x$ is injective, therefore monic. It is not split monic: if $em = 1$ then $e(2) = 1$, so $2e(1) = 1$, and there is no integer $x$ satisfying $2x = 1$.

(c) In **Top**, a map is monic iff injective (arguing as in 4.1.30(a)). A map $m : A \longrightarrow B$ is regular monic iff the induced map $A \longrightarrow m(A)$ is a homeomorphism. (So up to isomorphism, the regular monics are the inclusions of subspaces.) In particular, a bijection $m$ is regular monic if and only if it is a homeomorphism, so we get an example by writing down any example of a continuous bijection that is not a homeomorphism. For instance, let $A$ be $\mathbb{R}$ with the discrete topology, let $B$ be $\mathbb{R}$ with the usual topology, and let $m$ be the map that is the identity on underlying sets. Or let $A = [0, 1)$, let $B$ be the circle, thought of as consisting of the complex numbers of modulus 1, and put $m(t) = e^{2\pi i t}$.

## Sheet 8: Limits and colimits

1. Bookwork.

2. Definitions: bookwork. Second part is straight manipulation of definitions.

3. Choose a product cone on every pair $(B, C)$, with notation as in Sheet 7, q.2. For each $A, B, C$, define a function

$$\begin{array}{rccc} \alpha_{A,B,C} : & \mathcal{A}(A, B \times C) & \longrightarrow & \mathcal{A}(A, B) \times \mathcal{A}(A, C) \\ & \overline{q} & \longmapsto & (p_1^{B,C} \circ \overline{q},\ p_2^{B,C} \circ \overline{q}), \end{array}$$

which is bijective by definition of limit. Prove $\alpha$ natural.

4. 'Only if' is bookwork. For 'if', write $R$ for the right adjoint of $\Delta$. Let $D \in [\mathbb{I}, \mathcal{A}]$. Then $[\mathbb{I}, \mathcal{A}](\Delta A, D) \cong \mathcal{A}(A, R(D))$ naturally in $A \in \mathcal{A}$. Applying 4.4.2, conclude that $R(D)$ is a limit of $D$.

5.(a) Simplest of many possibilities: take the unique non-identity map in the category $\mathcal{A} = (\bullet \longrightarrow \bullet)$.

(b) Follows from observation that $X \overset{p}{\underset{p}{\rightrightarrows}} Y \xrightarrow{q} Z$ is a coequalizer if and only if $q$ is an isomorphism.

(c) Axiom of Choice (page 40) says exactly that epic $\Rightarrow$ split epic in **Set**. Then use dual of Sheet 7, q.5.

### Sheet 9: Limits and colimits of presheaves

1.(a) The meaning of 'computed pointwise' is the statement of Theorem 5.1.5 (with $\mathbb{A}$ changed to $\mathbb{A}^{\mathrm{op}}$ and $\mathbb{S}$ to **Set**).

(b) Applying Lemma 4.1.31, a map $\alpha$ in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ is monic iff a certain square involving $\alpha$ is a pullback, iff for each $A \in \mathbb{A}$ the analogous square involving $\alpha_A$ is a pullback (since pullbacks are computed pointwise), iff for each $A \in \mathbb{A}$ the map $\alpha_A$ is monic. The monics in **Set** are the injections, so $\alpha$ is monic iff each $\alpha_A$ is injective. Similarly, the epics are the pointwise surjections.

Without using (a), can still figure out what the monics are: do a direct proof by considering maps out of representables. But I know of no way of proving the result on epics without (a).

2. Bookwork.

3. Something stronger is true: every representable $H_C$ is **connected**, meaning that whenever $H_C \cong X + Y$ for presheaves $X$ and $Y$, then $X \cong 0$ or $Y \cong 0$. (Here $0 = \Delta\emptyset$ is the initial presheaf.) This implies the result in the question because $H_A \ncong 0$ (since we have $1_A \in H_A(A)$) and similarly $H_B \ncong 0$.

(Actually, connectedness also includes the condition of not being isomorphic to 0. This is very like the condition that 1 is not a prime number.)

To prove that $H_C$ is connected, suppose $H_C = X + Y$. Then have universal element $u \in (X+Y)(C) \cong X(C)+Y(C)$. Viewing $X(C)$ and $Y(C)$ as subsets of $(X+Y)(C)$, either $u \in X(C)$ or $u \in Y(C)$. If $u \in X(C)$ then $((X+Y)(f))(u) \in X(D)$ for all maps $D \xrightarrow{f} C$, which implies (by definition of universality) that $Y(D) = \emptyset$ for all $D$; hence $Y \cong 0$. Similarly, if $u \in Y(C)$ then $X \cong 0$.

4. Follows immediately from 3.3.2 and definition of $\mathbb{E}(X)$.

5.(a) If you're having trouble with 'only if', make sure you understand the definition of $\mathbf{Monic}(A)$; perhaps 2.3.3(a) will help. For 'if', write $I$ for the common image of $m$ and $m'$; then since monic = injective in **Set**, there is a bijection $j : X \longrightarrow I$ defined by $j(x) = m(x)$, and similarly $j' : X' \longrightarrow I$; show $(j')^{-1} \circ j$ is an isomorphism from $m$ to $m'$.

(b) Subgroups, subrings, vector subspaces. In **Top**, a subobject is a subset equipped with a topology containing the subspace topology. (If you'd prefer the answer to be 'subspaces', take *regular* subobjects instead: equivalence classes of regular monics. See Sheet 7, q.5(c).)

### Sheet 10: Interaction of (co)limits with adjunctions

1.(a) Bookwork.

(b) Given $A \in \mathcal{A}$, have to find left adjoint to $H^A : \mathcal{A} \longrightarrow \mathbf{Set}$. For $S \in \mathbf{Set}$ and $B \in \mathcal{A}$, a map $S \longrightarrow H^A(B)$ is a family $(A \xrightarrow{f_s} B)_{s \in S}$ of maps in $\mathcal{A}$, or equivalently a map $\sum_{s \in S} A \longrightarrow B$. So the left adjoint is $S \longmapsto \sum_{s \in S} A$. We usually write $\sum_{s \in S} A$ as $S \times A$ and call it a **copower** of $A$; compare powers (page 70). To explain the notation, if 2 is a two-element set then $2 \times A = A + A$, and similarly for other numbers. Also, if $\mathcal{A} = \mathbf{Set}$ then the copower $S \times A$ is the same as the product $S \times A$.

2.(a) Bookwork.

(b) $U$ does not preserve initial objects.

(c) $I$ does not preserve the sum $1 + 1$.
$C$ does not preserve the equalizer of the two distinct functors $\mathbf{1} \rightrightarrows \mathbf{2}$, where $\mathbf{2} = (\bullet \longrightarrow \bullet)$.

3. Bookwork.

4.(a) Straight application of definitions of pullback and monic.

(b) Just need to confirm that if $X_1 \xrightarrow{m_1} A$ and $X_2 \xrightarrow{m_2} A$ are monics representing same subobject of $A$ then the monics $X'_1 \xrightarrow{m'_1} A$ and $X'_2 \xrightarrow{m'_2} A$ obtained by pulling back along $f$ represent same subobject of $A'$. Can do this directly or prove a more general—and morally obvious—statement about isomorphic cones having isomorphic limits.

(c) Just need to check that Sub preserves identities (easy) and composition (direct from hint in question).

(d) Saw in Sheet 9, q.5 that in **Set**, subobjects are subsets. Saw in 4.1.16 that inverse images of subsets correspond to pullbacks of inclusions. From this, deduce that $\mathrm{Sub} \cong \mathcal{P}$, where $\mathcal{P}$ is as in 3.1.10(b). But saw there that $\mathcal{P} \cong H_2$, so $\mathrm{Sub} \cong H_2$.