The eventual image

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1. Overview

One-slide summary of talk

Question: Is there a universal method for turning an *endomorphism* of an object into an *automorphism* of a (perhaps different) object?

Answer: Yes, for trivial general reasons. There are both left and right universal methods.

Surprise: When the objects concerned are 'finite', the left and right universal methods are the same.

More precise one-slide summary

Let $\mathscr{C}$ be a category.

Write Endo($\mathscr{C}$) for the category in which:

- objects are pairs (X, T) with $X \in \mathscr{C}$ and $T\colon X \longrightarrow X$
- maps $(X, T) \longrightarrow (Y, S)$ are maps $f\colon X \longrightarrow Y$ such that $Sf = fT$.

Write Auto($\mathscr{C}$) for the full subcategory of objects (X, T) where T is an automorphism.

For general reasons, $\mathrm{Auto}(\mathscr{C}) \hookrightarrow \mathrm{Endo}(\mathscr{C})$ usually has both adjoints.

The surprise: When the objects of $\mathscr{C}$ are 'finite', these adjoints are equal.

Examples

Let $\mathscr{C}$ be any of these categories:

- **FinSet** = finite sets
- **FDVS** = finite-dimensional vector spaces
- **CptMS** = compact metric spaces and distance-decreasing maps.

Then:

$\mathrm{Auto}(\mathscr{C}) \hookrightarrow \mathrm{Endo}(\mathscr{C})$ has a simultaneous left and right adjoint,

$$\begin{array}{ccc} \mathrm{Endo}(\mathscr{C}) & \longrightarrow & \mathrm{Auto}(\mathscr{C}) \\ (X, T) & \longmapsto & (\boxed{\mathrm{im}^\infty(T)}, T'). \end{array}$$

the eventual image of T

Moreover, for all $(X, T) \in \mathrm{Endo}(\mathscr{C})$, the composite

$$(\mathrm{im}^\infty(T), T') \xrightarrow{\text{counit}} (X, T) \xrightarrow{\text{unit}} (\mathrm{im}^\infty(T), T')$$

is the identity.

$\mathscr{C}$ has eventual images

Eventual images, explicitly

Let T be an endomorphism of an object X.

An eventual image of T is a retract

$$\begin{array}{c} (X, T) \circlearrowleft T^{\infty} \\ \uparrow \quad \downarrow \\ (\mathrm{im}^{\infty}(T), T') \end{array}$$

of (X, T) such that T' is an automorphism and:

- whenever S is an automorphism of an object Y, any map $(Y, S) \longrightarrow (X, T)$ factors uniquely through $(\mathrm{im}^{\infty}(T), T') \rightarrowtail (X, T)$;
- whenever S is an automorphism of an object Y, any map $(X, T) \longrightarrow (Y, S)$ factors uniquely through $(X, T) \twoheadrightarrow (\mathrm{im}^{\infty}(T), T')$.

Composing gives an idempotent T^{∞} on X.

Plan for rest of talk

2. Sets

3. Vector spaces

4. Metric spaces

5. Unifying theorem

2. Sets

Sets

Let X be a finite set and $T\colon X \longrightarrow X$. Then

$$X \supseteq TX \supseteq T^2X \supseteq \cdots .$$

Put $\mathrm{im}^{\infty}(T) = \bigcap_{n\in\mathbb{N}} T^nX = \bigcap_{n\in\mathbb{N}} \mathrm{im}(T^n)$.

Lemma

T restricts to an automorphism T' of $\mathrm{im}^{\infty}(T)$.

This gives

$$\begin{array}{c}(X, T)\\ \uparrow\\ (\mathrm{im}^{\infty}(T), T').\end{array}$$

Sets

Lemma

$\{T, T^2, T^3, \ldots\}$ *contains a unique idempotent,* T^∞*, say.*

Lemma

$\mathrm{im}(T^\infty) = \mathrm{im}^\infty(T)$*.*

This gives maps

$$\begin{array}{c} (X, T) \circlearrowleft T^\infty \\ \uparrow \quad \downarrow \\ (\mathrm{im}^\infty(T), T') \end{array}$$

with the universal properties required.

3. Vector spaces

Vector spaces

Let X be a finite-dimensional vector space and $T\colon X \longrightarrow X$. Put

$$\mathrm{im}^{\infty}(T) = \bigcap_{n\in\mathbb{N}} \mathrm{im}(T^n), \qquad \ker^{\infty}(T) = \bigcup_{n\in\mathbb{N}} \ker(T^n).$$

Lemma

T restricts to an automorphism T' of $\mathrm{im}^{\infty}(T)$.

Lemma

$X = \mathrm{im}^{\infty}(T) \oplus \ker^{\infty}(T)$.

This decomposition gives projection and inclusion maps

$$\begin{array}{c} (X, T) \circlearrowleft T^{\infty} \\ \uparrow\ \downarrow \\ (\mathrm{im}^{\infty}(T), T') \end{array}$$

with the universal properties required.

(In fact, T^{∞} is a polynomial in T.)

4. Metric spaces

Metric spaces

Consider category of compact metric spaces and distance-decreasing maps $(d(f(x_1), f(x_2)) \leqslant d(x_1, x_2))$.

Background fact: For a map $T\colon X \longrightarrow X$ in this category,

$$\begin{aligned} T \text{ is distance-preserving} &\iff T \text{ is invertible} \\ &\iff T \text{ is surjective.} \end{aligned}$$

Let $T\colon X \longrightarrow X$ be an endomorphism. Put $\mathrm{im}^\infty(T) = \bigcap_{n\in\mathbb{N}} \mathrm{im}(T^n)$.

Lemma

T restricts to an automorphism T' of $\mathrm{im}^\infty(T)$.

This gives

$$\begin{array}{c} (X, T) \\ \uparrow \\ (\mathrm{im}^\infty(T), T'). \end{array}$$

Metric spaces

Lemma

The inclusion

$$\begin{array}{c} (X, T) \\ \big\uparrow \\ (\mathrm{im}^\infty(T), T'). \end{array}$$

has a canonical retraction.

The maps

$$\begin{array}{c} (X, T) \circlearrowleft T^\infty \\ \big\uparrow \;\; \big\downarrow \\ (\mathrm{im}^\infty(T), T') \end{array}$$

have the universal properties required.

(In fact, T^∞ is in the closure of $\{T, T^2, \ldots\}$, for a suitable topology.)

5. Unifying theorem

The theorem

Let $\mathscr{C}$ be a category with a factorization system. Call the left maps surjections ($\twoheadrightarrow$) and the right maps embeddings ($\rightarrowtail$), and suppose that:

- for an endomorphism T in $\mathscr{C}$,

$$\begin{aligned} T \text{ is an embedding} &\iff T \text{ is invertible} \\ &\iff T \text{ is a surjection;} \end{aligned}$$

- limits of sequences $\cdots \rightarrowtail \cdot \rightarrowtail \cdot$ and colimits of sequences $\cdot \twoheadrightarrow \cdot \twoheadrightarrow \cdots$ exist and preserve factorizations.

Theorem

$\mathscr{C}$ has eventual images. In particular, $\mathrm{Auto}(\mathscr{C}) \hookrightarrow \mathrm{Endo}(\mathscr{C})$ *has a simultaneous left and right adjoint.*

Proof: Show that $\mathrm{im}^\infty(T)$ is limit *and* colimit of $\cdots \xrightarrow{T} X \xrightarrow{T} X \xrightarrow{T} \cdots$.

Examples of the theorem

$\mathscr{C}$	embeddings	surjections
FinSet	injections	surjections
FDVS	injections	surjections
CptMS	distance-preserving maps	surjections
$\mathbf{FDVS}^G$ (group rep'ns)	injections	surjections
FinGrp, **FinRing**, etc.	injections	surjections

Special properties of the eventual image

Proposition

Eventual image is tracelike: given maps

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y,$$

we have $\mathrm{im}^\infty(gf) \cong \mathrm{im}^\infty(fg)$*.*

Proposition

Eventual image is dynamical: given $T \colon X \longrightarrow X$*, we have*

$$\mathrm{im}^\infty(T) = \mathrm{im}^\infty(T^2) = \mathrm{im}^\infty(T^3) = \cdots .$$

Open questions

1. Is there a more satisfactory general (enriched) setting?

2. When is a left Kan extension equal to a right Kan extension?

   (The inclusion $i \colon \mathbb{N} \longrightarrow \mathbb{Z}$ of additive monoids induces a functor $[\mathbb{Z}, \mathscr{C}] \longrightarrow [\mathbb{N}, \mathscr{C}]$, which is just $\mathrm{Auto}(\mathscr{C}) \hookrightarrow \mathrm{Endo}(\mathscr{C})$. If $\mathscr{C}$ has eventual images then left and right Kan extensions along i are equal.)

3. What about discrete-time dynamical systems in less finite contexts?

   E.g. for the endomorphism $z \mapsto z^2$ of $\mathbb{C} \cup \{\infty\}$, the eventual image should probably be $\{z \in \mathbb{C} : |z| = 1\} \cup \{0, \infty\}$.