The Euler characteristic of an associative algebra

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joint with

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and inspired by work of

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Preview of the main theorem

Theorem

Let A be an algebra, of finite dimension and finite global dimension, over an algebraically closed field.

Then the magnitude of the **Vect**-category of projective indecomposable A-modules is equal to

$$\chi_A(S, S),$$

where

- χ_A is the Euler form of A;
- S is the direct sum of the simple A-modules (one per iso class).

But first, I will:

- define the terms in red (and some of the others);
- explain why you might care.

Plan

1. The magnitude of an enriched category

2. Algebraic background

3. The theorem

1. The magnitude of an enriched category

The definition

Let $(\mathbf{V}, \otimes, I)$ be a monoidal category.

Idea: Notion of size for $\mathbf{V}$-objects $\mapsto$ notion of size for $\mathbf{V}$-categories.

Suppose we have a semiring R and a monoid homomorphism

$$|\cdot|\colon (\mathrm{ob}\,\mathbf{V}/\cong, \otimes, I) \longrightarrow (R, \cdot, 1).$$

(E.g. $\mathbf{V} = \mathbf{FinSet}$, $R = \mathbb{Q}$, $|\cdot| =$ cardinality.)

Let $\mathbf{A}$ be a $\mathbf{V}$-category with finite object-set. Define an $\mathrm{ob}\,\mathbf{A} \times \mathrm{ob}\,\mathbf{A}$ matrix

$$Z_{\mathbf{A}} = \big(|\mathbf{A}(a, b)|\big)_{a,b \in A}$$

over R.

Assuming $Z_{\mathbf{A}}$ is invertible, the magnitude of $\mathbf{A}$ is

$$|\mathbf{A}| = \sum_{a,b \in \mathbf{A}} (Z_{\mathbf{A}}^{-1})(a, b) \in R.$$

Ordinary categories

Let $\mathbf{V} = \mathbf{FinSet}$, $R = \mathbb{Q}$, and $|X| = \mathrm{card}(X)$.

We obtain a notion of the magnitude $|\mathbf{A}| \in \mathbb{Q}$ of a finite category $\mathbf{A}$.

Example: If $\mathbf{A}$ is discrete then $|\mathbf{A}| = \mathrm{card}(\mathrm{ob}\,\mathbf{A})$.

Example: Every small category $\mathbf{A}$ has a classifying space $B\mathbf{A} \in \mathbf{Top}$.

And assuming certain finiteness hypotheses,

$$|\mathbf{A}| = \chi(B\mathbf{A}).$$

For this reason, $|\mathbf{A}|$ is also called the Euler characteristic of $\mathbf{A}$.

Fundamental idea (Schanuel):

Euler characteristic is the topological analogue of cardinality.

Ordered sets

Let $\mathbf{V} = \mathbf{2} = \{\text{false} < \text{true}\}$. Let $R = \mathbb{Z}$, $|\text{false}| = 0$ and $|\text{true}| = 1$.

We obtain a notion of the magnitude $|P| \in \mathbb{Z}$ of a finite poset P.

(It is equal to what is called $\chi(P)$ in poset homology.)

The matrix Z_P^{-1} is the 'Möbius function' of P (Rota et al—combinatorics).

Example: If $(P, \leq) = (\mathbb{N}, |\,)$ (not finite, but never mind...) then

$$Z_P^{-1}(a, b) = \begin{cases} \mu(b/a) & \text{if } a|b \\ 0 & \text{otherwise} \end{cases}$$

where μ is the classical number-theoretic Möbius function.

Metric spaces

Let $\mathbf{V} = ([0, \infty], \geq)$, $\otimes = +$ and $I = 0$. Let $R = \mathbb{R}$ and $|x| = e^{-x}$ (why? so that $x \mapsto |x|$ is a monoid homomorphism).

We obtain a notion of the magnitude $|A| \in \mathbb{R}$ of a finite metric space A.

The definition extends to compact subsets $A \subseteq \mathbb{R}^n$.

It is geometrically informative. For example:

Theorem (Meckes): Let $A \subseteq \mathbb{R}^n$ be compact. The asymptotic growth of the function $t \mapsto |tA|$ is equal to the Minkowski dimension of A.

Conjecture (with Willerton): Let $A \subseteq \mathbb{R}^2$ be compact convex. For $t > 0$,

$$|tA| = \chi(A) + \frac{\text{perimeter}(A)}{4} \cdot t + \frac{\text{area}(A)}{2\pi} \cdot t^2.$$

Linear categories

Let $\mathbf{V} = \mathbf{FDVect}$, $R = \mathbb{Q}$, and $|X| = \dim X$.

We obtain a notion of the magnitude $|\mathbf{A}|$ of a finite linear category ($\mathbf{V}$-category).

Our main theorem will provide an example...

2. Algebraic background

Conventions

Throughout, we fix:

- an algebraically closed field K;
- a finite-dimensional associative K-algebra A (unital, but not necessarily commutative).

We will consider finite-dimensional A-modules.

The atoms of the module world

Question: Which A-modules deserve to be thought of as 'atomic'?

Answer 1: The simple modules.
(A module is simple if it is nonzero and has no nontrivial submodules.)

Some facts about simple modules:

- There are only finitely many (up to isomorphism), and they are finite-dimensional.
- If S and T are simple then

$$\mathrm{Hom}_A(S, T) \cong \begin{cases} K & \text{if } S \cong T \\ 0 & \text{otherwise.} \end{cases}$$

The atoms of the module world

Question: Which A-modules deserve to be thought of as 'atomic'?

Answer 2: The projective indecomposable modules.
(A module P is projective if $\mathrm{Hom}_A(P, -)$ preserves epimorphisms, and indecomposable if it is nonzero and has no nontrivial direct summands.)

Some facts about projective indecomposables:

- There are only finitely many (up to isomorphism), and they are finite-dimensional.
  So the linear category $\mathbf{ProjIndec}(A)$ of projective indecomposable modules is essentially finite.
- The A-module A is a direct sum of projective indecomposable modules. Every projective indecomposable appears at least once in this sum.
- The linear category $\mathbf{ProjIndec}(A)$ has the same representations as the algebra A. That is,

$$[\mathbf{ProjIndec}(A), \mathbf{Vect}] \simeq A\text{-}\mathbf{Mod}.$$

The atoms of the module world

How do these two answers compare?

$$\text{Simple} \;\begin{array}{c}\not\Longrightarrow \\ \not\Longleftarrow\end{array}\; \text{projective indecomposable.}$$

But there is a natural bijection

$$\{\text{simple modules}\}/\cong \quad \longleftrightarrow \quad \{\text{projective indecomposables}\}/\cong$$

given by $S \leftrightarrow P$ iff S is a quotient of P.

(It is not an equivalence of categories!)

Choose representative families

$(S_i)_{i \in I}$ of the iso classes of simple modules,
$(P_i)_{i \in I}$ of the iso classes of projective indecomposable modules,

with S_i a quotient of P_i.

Ext and the Euler form

For each $n \geq 0$, we have the functor

$$\mathrm{Ext}_A^n \colon A\text{-}\mathbf{Mod}^{\mathrm{op}} \times A\text{-}\mathbf{Mod} \longrightarrow \mathbf{Vect}.$$

Assume now that A has finite global dimension, i.e. $\mathrm{Ext}_A^n = 0$ for all $n \gg 0$.

For finite-dimensional modules X and Y, define

$$\chi_A(X, Y) = \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}_A^n(X, Y) \in \mathbb{Z}.$$

Algebraists call χ_A the Euler form of A. It is biadditive.

Remark: Writing $S = \bigoplus_{i \in I} S_i$, we have

$$\chi_A(S, S) = \sum_{i,j \in I} \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}_A^n(S_i, S_j).$$

And although $\mathrm{Hom}_A(S_i, S_j)$ is trivial, $\mathrm{Ext}_A^n(S_i, S_j)$ is interesting.

3. The theorem

Statement of the theorem (again)

Recall: A is an algebra, of finite dimension and finite global dimension, over an algebraically closed field. We write:

- **ProjIndec**(A) for the linear category of projective indecomposable A-modules, which is essentially finite;
- $|\mathbf{A}|$ for the magnitude of an (enriched) category $\mathbf{A}$;
- $S = \bigoplus_{i \in I} S_i$, where $(S_i)_{i \in I}$ is a representative family of the isomorphism classes of simple A-modules.

Theorem $|\mathbf{ProjIndec}(A)| = \chi_A(S, S)$.

Explicitly, this means: define a matrix

$$Z_A = \big(\dim \mathrm{Hom}_A(P_i, P_j)\big)_{i,j \in I}.$$

Then

$$\sum_{i,j \in I} (Z_A^{-1})(i,j) = \sum_{i,j \in I} \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}_A^n(S_j, S_i).$$

Examples

Example (Stroppel): Let A be a Koszul algebra. Then A is naturally graded, and $S = A_0$. Hence

$$|\mathbf{ProjIndec}(A)| = \sum_{n=0}^{\infty} (-1)^n \dim \operatorname{Ext}_A^n(A_0, A_0).$$

Example: Let $(Q_1 \rightrightarrows Q_0)$ be a finite acyclic quiver (directed graph).

Take its path algebra A.

The simple/projective indecomposable modules are indexed by the vertex-set Q_0, and one can calculate homologically that

$$\chi_A(S, S) = \operatorname{card}(Q_0) - \operatorname{card}(Q_1).$$

Hence

$$|\mathbf{ProjIndec}(A)| = \operatorname{card}(Q_0) - \operatorname{card}(Q_1).$$

Proof of the theorem: the strategy

The theorem is proved by moving between several different descriptions of the matrix Z_A:

- $Z_A(i,j) = \dim \operatorname{Hom}_A(P_i, P_j)$ (by definition).
- $Z_A(i,j) = \chi_A(P_i, P_j)$.
- $Z_A(i,j)$ is the multiplicity of S_i as a composition factor of P_j (in the jargon: Z_A is the Cartan matrix of A).
- Both $(S_i)_{i \in I}$ and $(P_i)_{i \in I}$ are bases of the Grothendieck group of finite-dimensional A-modules, and Z_A is the change-of-basis matrix.

Conclusion

What is the right definition of the Euler characteristic of an algebra A?

A category theorist's answer:

- Schanuel taught us: Euler characteristic is the canonical measure of size.
- There is a general definition of the magnitude/Euler characteristic/size of an enriched category.
- An important enriched category associated with A is **ProjIndec**(A).
- So, define the Euler characteristic of A as the magnitude of **ProjIndec**(A).

An algebraist's answer:

- We know the importance of the Euler *form* of A, defined by a homological formula: $\chi_A(-,-) = \sum(-1)^n \dim \mathrm{Ext}^n_A(-,-)$.
- We know the importance of the simple modules, and their direct sum S.
- So, define the Euler characteristic of A as $\chi_A(S, S)$.

The theorem states that the two answers are the same.