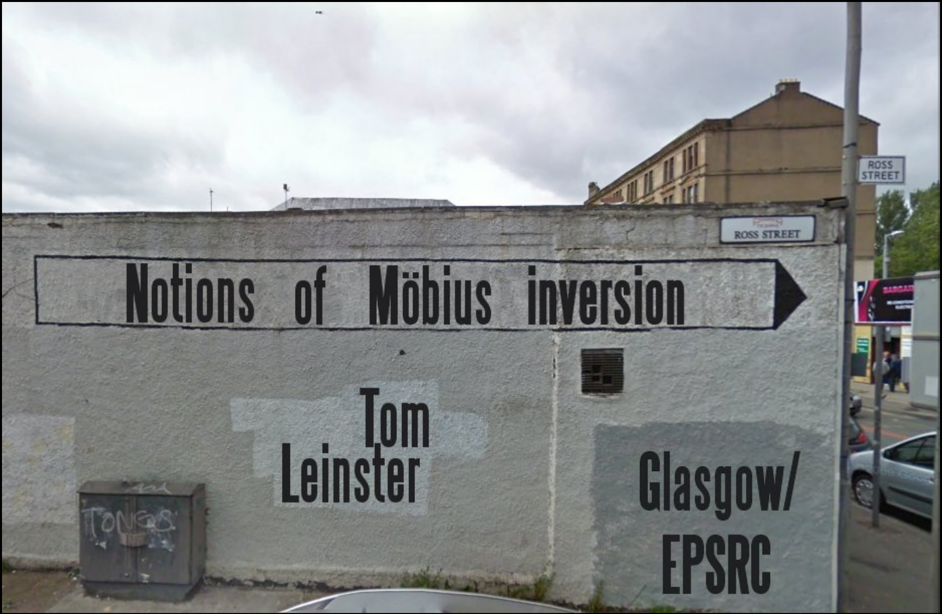
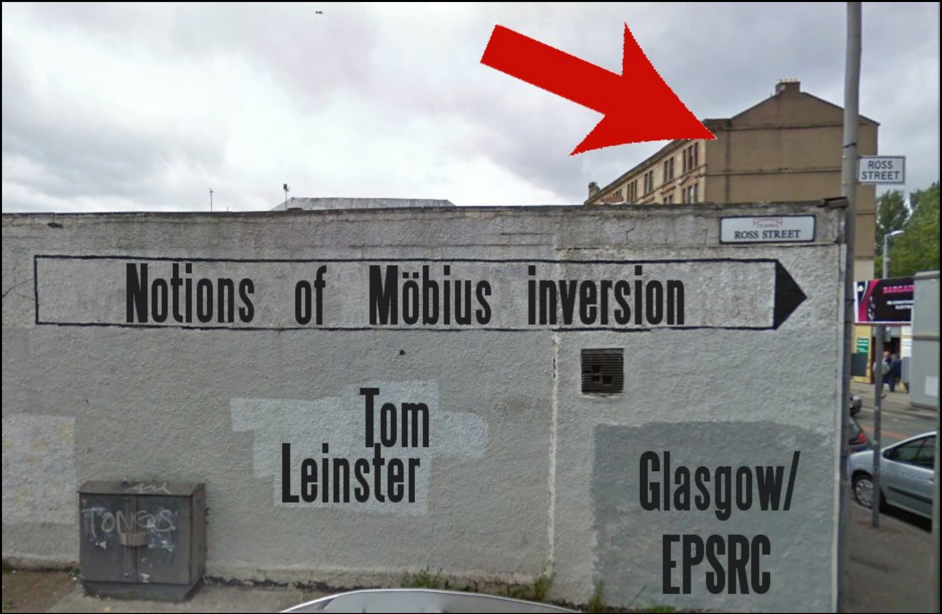
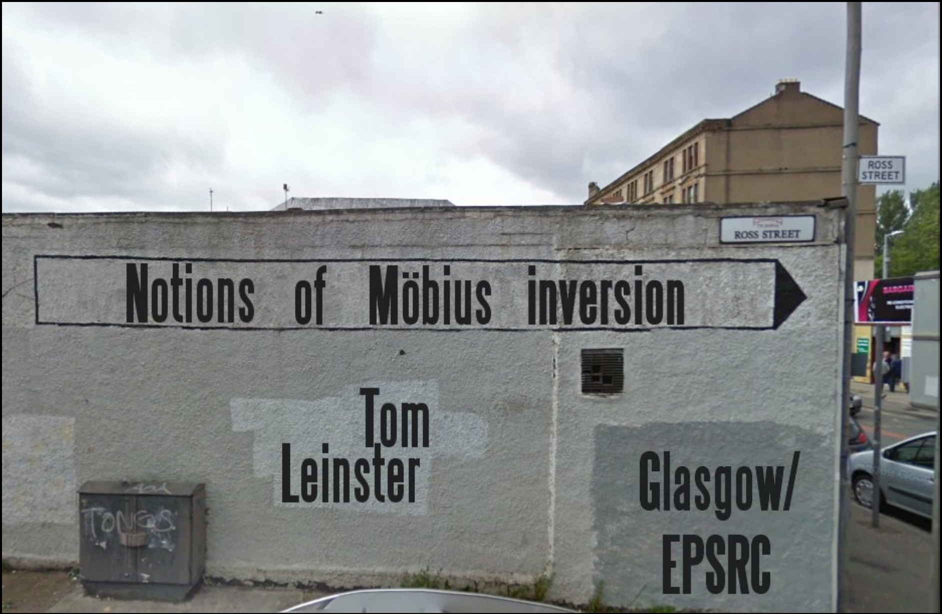

Notions of Möbius inversion
Tom
Leinster
Glasgow/
EPSRC
ROSS STREET
ROSS
STREET


ROSS STREET
ROSS STREET
Notions of Möbius inversion
Tom
Leinster
Glasgow/
EPSRC


ROSS STREET
ROSS STREET
Notions of Möbius inversion
Tom Leinster
Glasgow/ EPSRC


SMALL SHOE CO.
LADIES 1. 1½.
SIZES 2. 2½..
ROSS STREET
ROSS STREET

Prelude:
How important is composition in a category?

Prelude:
How important is composition in a category?

How important is the topology of a topological space?

Prelude:
How important is composition in a category?

How important is the topology of a topological space?

- A cell complex is a gluing of balls: 

Prelude:
How important is composition in a category?

How important is the topology of a topological space?

- A cell complex is a gluing of balls: 
- Its topology depends entirely on how they are glued.

Prelude:
How important is composition in a category?

How important is the topology of a topological space?

- A cell complex is a gluing of balls:  $\cdots$
- Its topology depends entirely on how they are glued.
- But its Euler characteristic does not.

Prelude:
How important is composition in a category?

How important is the topology of a topological space?

- A cell complex is a gluing of balls: 
- Its topology depends entirely on how they are glued.
- But its Euler characteristic does not.

Let $\mathbf{A}$ be a small category. Write $B\mathbf{A} = |N\mathbf{A}|$ for its classifying space.

Prelude:
How important is composition in a category?

How important is the topology of a topological space?

- A cell complex is a gluing of balls: 
- Its topology depends entirely on how they are glued.
- But its Euler characteristic does not.

Let $\mathbf{A}$ be a small category. Write $B\mathbf{A} = |N\mathbf{A}|$ for its classifying space.

Suppose that $\mathbf{A}$ is suitably finite, so that $\chi(B\mathbf{A})$ is defined.

Prelude:
How important is composition in a category?

How important is the topology of a topological space?

- A cell complex is a gluing of balls: 
- Its topology depends entirely on how they are glued.
- But its Euler characteristic does not.

Let $\mathbf{A}$ be a small category. Write $B\mathbf{A} = |N\mathbf{A}|$ for its classifying space.

Suppose that $\mathbf{A}$ is suitably finite, so that $\chi(B\mathbf{A})$ is defined.

Theorem

$\chi(B\mathbf{A})$ *is independent of the composition and identities in* $\mathbf{A}$.

Prelude:
How important is composition in a category?

How important is the topology of a topological space?

- A cell complex is a gluing of balls: 
- Its topology depends entirely on how they are glued.
- But its Euler characteristic does not.

Let $\mathbf{A}$ be a small category. Write $B\mathbf{A} = |N\mathbf{A}|$ for its classifying space.

Suppose that $\mathbf{A}$ is suitably finite, so that $\chi(B\mathbf{A})$ is defined.

Theorem

$\chi(B\mathbf{A})$ *is independent of the composition and identities in* $\mathbf{A}$*.*

That is, if $\mathbf{A}$ and $\mathbf{A}'$ have the same underlying graph then $\chi(B\mathbf{A}) = \chi(B\mathbf{A}')$.

Plan

Plan

1. A simplified history of Möbius inversion

Plan

1. A simplified history of Möbius inversion

2. Fine vs. coarse Möbius inversion

Plan

1. A simplified history of Möbius inversion
2. Fine vs. coarse Möbius inversion
3. How important is composition in a category?

Plan

1. A simplified history of Möbius inversion
2. Fine vs. coarse Möbius inversion
3. How important is composition in a category?
4. Postscript? A theorem of Beck–Chevalley type

1. A simplified history of Möbius inversion

Overview

Overview

Number-theoretic Möbius inversion
(Möbius 1832)

Overview

Number-theoretic Möbius inversion
(Möbius 1832)

↓

Möbius inversion for posets
(Rota 1964, et al.)

Overview

Number-theoretic Möbius inversion
(Möbius 1832)

↓

Möbius inversion for posets
(Rota 1964, et al.)

↙

Fine M. inversion for categories
(Leroux et al. 1975, 1980;
Haigh 1980)

Overview

Number-theoretic Möbius inversion
(Möbius 1832)

Möbius inversion for posets
(Rota 1964, et al.)

Fine M. inversion for categories
(Leroux et al. 1975, 1980;
Haigh 1980)

Coarse M. inversion for categories
(Haigh 1980;
Leinster 2008)

Number-theoretic Möbius inversion

Number-theoretic Möbius inversion

For $\alpha, \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$, define $\alpha * \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by

$$(\alpha * \beta)(n) = \sum_{km=n} \alpha(k)\beta(m).$$

Number-theoretic Möbius inversion

For $\alpha, \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$, define $\alpha * \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by

$$(\alpha * \beta)(n) = \sum_{km=n} \alpha(k)\beta(m).$$

The unit for the product $*$ is $\delta \colon \mathbb{Z}^+ \to \mathbb{Z}$, given by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Number-theoretic Möbius inversion

For $\alpha, \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$, define $\alpha * \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by

$$(\alpha * \beta)(n) = \sum_{km=n} \alpha(k)\beta(m).$$

The unit for the product $*$ is $\delta \colon \mathbb{Z}^+ \to \mathbb{Z}$, given by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by $\zeta(n) = 1$ for all n.

Number-theoretic Möbius inversion

For $\alpha, \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$, define $\alpha * \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by

$$(\alpha * \beta)(n) = \sum_{km=n} \alpha(k)\beta(m).$$

The unit for the product $*$ is $\delta \colon \mathbb{Z}^+ \to \mathbb{Z}$, given by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by $\zeta(n) = 1$ for all n.

The Möbius function $\mu \colon \mathbb{Z}^+ \to \mathbb{Z}$ is the inverse ζ^{-1} of ζ.

Number-theoretic Möbius inversion

For $\alpha, \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$, define $\alpha * \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by

$$(\alpha * \beta)(n) = \sum_{km=n} \alpha(k)\beta(m).$$

The unit for the product $*$ is $\delta \colon \mathbb{Z}^+ \to \mathbb{Z}$, given by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by $\zeta(n) = 1$ for all n.

The Möbius function $\mu \colon \mathbb{Z}^+ \to \mathbb{Z}$ is the inverse ζ^{-1} of ζ. Explicitly,

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Number-theoretic Möbius inversion

For $\alpha, \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$, define $\alpha * \beta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by

$$(\alpha * \beta)(n) = \sum_{km=n} \alpha(k)\beta(m).$$

The unit for the product $*$ is $\delta \colon \mathbb{Z}^+ \to \mathbb{Z}$, given by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \colon \mathbb{Z}^+ \to \mathbb{Z}$ by $\zeta(n) = 1$ for all n.

The Möbius function $\mu \colon \mathbb{Z}^+ \to \mathbb{Z}$ is the inverse ζ^{-1} of ζ. Explicitly,

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Important in number theory, e.g.

$$1 \Big/ \sum_n \frac{1}{n^s} = \sum_n \frac{\mu(n)}{n^s}.$$

Möbius inversion for posets

Möbius inversion for posets

Let A be a suitably finite poset. Let k be a (commutative) ring.

Möbius inversion for posets

Let A be a suitably finite poset. Let k be a (commutative) ring.

The incidence algebra kA is the set of functions

$$\{(a, b) \in A \times A \mid a \leq b\} \to k,$$

Möbius inversion for posets

Let A be a suitably finite poset. Let k be a (commutative) ring.

The incidence algebra kA is the set of functions

$$\{(a, b) \in A \times A \mid a \leq b\} \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a, c) = \sum_{b\colon\, a \leq b \leq c} \alpha(a, b)\, \beta(b, c)$$

$(\alpha, \beta \in kA)$,

Möbius inversion for posets

Let A be a suitably finite poset. Let k be a (commutative) ring.

The incidence algebra kA is the set of functions

$$\{(a, b) \in A \times A \mid a \leq b\} \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a, c) = \sum_{b\colon\ a \leq b \leq c} \alpha(a, b)\, \beta(b, c)$$

($\alpha, \beta \in kA$), and unit δ given by

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Möbius inversion for posets

Let A be a suitably finite poset. Let k be a (commutative) ring.

The incidence algebra kA is the set of functions

$$\{(a,b) \in A \times A \mid a \leq b\} \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a,c) = \sum_{b\colon\, a \leq b \leq c} \alpha(a,b)\,\beta(b,c)$$

($\alpha, \beta \in kA$), and unit δ given by

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in kA$ by $\zeta(a,b) = 1$ for all a, b.

Möbius inversion for posets

Let A be a suitably finite poset. Let k be a (commutative) ring.

The incidence algebra kA is the set of functions

$$\{(a,b) \in A \times A \mid a \leq b\} \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a,c) = \sum_{b\colon a \leq b \leq c} \alpha(a,b)\,\beta(b,c)$$

($\alpha, \beta \in kA$), and unit δ given by

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in kA$ by $\zeta(a,b) = 1$ for all a, b.

The Möbius function μ is ζ^{-1}. (It always exists.)

Möbius inversion for posets

Let A be a suitably finite poset. Let k be a (commutative) ring.

The incidence algebra kA is the set of functions

$$\{(a, b) \in A \times A \mid a \leq b\} \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a, c) = \sum_{b:\ a \leq b \leq c} \alpha(a, b)\, \beta(b, c)$$

($\alpha, \beta \in kA$), and unit δ given by

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in kA$ by $\zeta(a, b) = 1$ for all a, b.

The Möbius function μ is ζ^{-1}. (It always exists.) Explicitly,

$$\mu(a, b) = \sum_{n \in \mathbb{N}} (-1)^n |\{\text{chains } a = a_0 < \cdots < a_n = b\}|.$$

Möbius inversion for posets

Let A be a suitably finite poset. Let k be a (commutative) ring.

The incidence algebra kA is the set of functions

$$\{(a,b) \in A \times A \mid a \leq b\} \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a,c) = \sum_{b:\ a \leq b \leq c} \alpha(a,b)\,\beta(b,c)$$

($\alpha, \beta \in kA$), and unit δ given by

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in kA$ by $\zeta(a,b) = 1$ for all a, b.

The Möbius function μ is ζ^{-1}. (It always exists.) Explicitly,

$$\mu(a,b) = \sum_{n \in \mathbb{N}} (-1)^n |\{\text{chains } a = a_0 < \cdots < a_n = b\}|.$$

E.g.: $(A, \leq) = (\mathbb{Z}^+, |)$: then $\mu(a,b) = \mu_{\text{classical}}(b/a)$.

Fine Möbius inversion for categories

Fine Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

Fine Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\text{arr}(\mathbf{A}) \to k,$$

Fine Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\mathrm{arr}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h\colon\ hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$,

Fine Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\mathrm{arr}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h\colon\, hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Fine Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\mathrm{arr}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h:\ hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

Fine Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\mathrm{arr}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h\colon\, hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

The fine Möbius function μ is ζ^{-1}, if it exists. Then $\mathbf{A}$ has fine Möbius inversion.

Fine Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\mathrm{arr}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h:\ hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

The fine Möbius function μ is ζ^{-1}, if it exists. Then $\mathbf{A}$ has fine Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$.

Fine Möbius inversion for categories

Let **A** be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\mathrm{arr}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h:\ hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

The fine Möbius function μ is ζ^{-1}, if it exists. Then **A** has fine Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA = k\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k\mathbf{A}$.

Coarse Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\mathrm{arr}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h:\ hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

The fine Möbius function μ is ζ^{-1}, if it exists. Then $\mathbf{A}$ has fine Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA = k\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k\mathbf{A}$.

Coarse Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The fine incidence algebra $k\mathbf{A}$ is the set of functions

$$\mathrm{arr}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h:\ hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

The fine Möbius function μ is ζ^{-1}, if it exists. Then $\mathbf{A}$ has fine Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA = k\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k\mathbf{A}$.

Coarse Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The coarse incidence algebra $k_c\mathbf{A}$ is the set of functions

$$\mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k,$$

---

with multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{g,h:\ hg=f} \alpha(g)\beta(h)$$

$(\alpha, \beta \in k\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

The fine Möbius function μ is ζ^{-1}, if it exists. Then $\mathbf{A}$ has fine Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA = k\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k\mathbf{A}$.

Coarse Möbius inversion for categories

Let **A** be a suitably finite category. Let k be a ring.

The coarse incidence algebra $k_c\mathbf{A}$ is the set of functions

$$\mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a, c) = \sum_b \alpha(a, b)\,\beta(b, c)$$

$(\alpha, \beta \in k_c\mathbf{A})$, and unit δ given by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

The fine Möbius function μ is ζ^{-1}, if it exists. Then **A** has fine Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA = k\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k\mathbf{A}$.

Coarse Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The coarse incidence algebra $k_c\mathbf{A}$ is the set of functions

$$\mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a, c) = \sum_b \alpha(a, b)\, \beta(b, c)$$

($\alpha, \beta \in k_c\mathbf{A}$), and unit δ given by

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k\mathbf{A}$ by $\zeta(f) = 1$ for all f.

The fine Möbius function μ is ζ^{-1}, if it exists. Then $\mathbf{A}$ has fine Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA = k\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k\mathbf{A}$.

Coarse Möbius inversion for categories

Let **A** be a suitably finite category. Let k be a ring.

The coarse incidence algebra $k_c\mathbf{A}$ is the set of functions

$$\mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a, c) = \sum_b \alpha(a, b)\, \beta(b, c)$$

($\alpha, \beta \in k_c\mathbf{A}$), and unit δ given by

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k_c\mathbf{A}$ by $\zeta(a, b) = |\mathrm{Hom}(a, b)|$.

The fine Möbius function μ is ζ^{-1}, if it exists. Then **A** has fine Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA = k\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k\mathbf{A}$.

Coarse Möbius inversion for categories

Let **A** be a suitably finite category. Let k be a ring.

The coarse incidence algebra $k_c\mathbf{A}$ is the set of functions

$$\mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a, c) = \sum_b \alpha(a, b)\, \beta(b, c)$$

($\alpha, \beta \in k_c\mathbf{A}$), and unit δ given by

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k_c\mathbf{A}$ by $\zeta(a, b) = |\mathrm{Hom}(a, b)|$.

The coarse Möbius function μ is ζ^{-1}, if it exists. Then **A** has coarse Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA = k\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k\mathbf{A}$.

Coarse Möbius inversion for categories

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

The coarse incidence algebra $k_c\mathbf{A}$ is the set of functions

$$\mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k,$$

with multiplication $*$ defined by

$$(\alpha * \beta)(a, c) = \sum_b \alpha(a, b)\, \beta(b, c)$$

$(\alpha, \beta \in k_c\mathbf{A})$, and unit δ given by

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Define $\zeta \in k_c\mathbf{A}$ by $\zeta(a, b) = |\mathrm{Hom}(a, b)|$.

The coarse Möbius function μ is ζ^{-1}, if it exists. Then $\mathbf{A}$ has coarse Möbius inversion.

E.g.: $\mathbf{A} = (A, \leq)$. Then $kA \subseteq k_c\mathbf{A}$, with $\zeta, \mu \in kA$ corresponding to $\zeta, \mu \in k_c\mathbf{A}$.

Fine and coarse: summary

Fine and coarse: summary

A: suitably finite category. k: ring.

Fine and coarse: summary

A: suitably finite category. k: ring.

	Fine	Coarse

Fine and coarse: summary

$\mathbf{A}$: suitably finite category. k: ring.

	Fine	Coarse
Incidence algebra	$k\mathbf{A} = \{\text{fns arr}(\mathbf{A}) \to k\}$	$k_c\mathbf{A} = \{\text{fns ob}(\mathbf{A}) \times \text{ob}(\mathbf{A}) \to k\}$

Fine and coarse: summary

A: suitably finite category. k: ring.

	Fine	Coarse
Incidence algebra	$k\mathbf{A} = \{\text{fns } \mathrm{arr}(\mathbf{A}) \to k\}$	$k_{\mathrm{c}}\mathbf{A} = \{\text{fns } \mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k\}$
Multiplication	$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\beta(h)$	$(\alpha * \beta)(a, b) = \sum_{b} \alpha(a, b)\beta(b, c)$

Fine and coarse: summary

A: suitably finite category. k: ring.

	Fine	Coarse
Incidence algebra	$k\mathbf{A} = \{\text{fns } \mathrm{arr}(\mathbf{A}) \to k\}$	$k_c\mathbf{A} = \{\text{fns } \mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k\}$
Multiplication	$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\beta(h)$	$(\alpha * \beta)(a, b) = \sum_{b} \alpha(a, b)\beta(b, c)$
Zeta function	$\zeta(f) \equiv 1$	$\zeta(a, b) = \lvert\mathrm{Hom}(a, b)\rvert$

Fine and coarse: summary

A: suitably finite category. k: ring.

	Fine	Coarse
Incidence algebra	$k\mathbf{A} = \{\text{fns } \mathrm{arr}(\mathbf{A}) \to k\}$	$k_{\mathrm{c}}\mathbf{A} = \{\text{fns } \mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k\}$
Multiplication	$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\beta(h)$	$(\alpha * \beta)(a, b) = \sum_{b} \alpha(a, b)\beta(b, c)$
Zeta function	$\zeta(f) \equiv 1$	$\zeta(a, b) = \lvert\mathrm{Hom}(a, b)\rvert$
Inverse of ζ	fine Möbius function μ	coarse Möbius function μ

Fine and coarse: summary

A: suitably finite category. k: ring.

	Fine	Coarse
Incidence algebra	$k\mathbf{A} = \{\text{fns arr}(\mathbf{A}) \to k\}$	$k_c\mathbf{A} = \{\text{fns ob}(\mathbf{A}) \times \text{ob}(\mathbf{A}) \to k\}$
Multiplication	$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\beta(h)$	$(\alpha * \beta)(a, b) = \sum_b \alpha(a, b)\beta(b, c)$
Zeta function	$\zeta(f) \equiv 1$	$\zeta(a, b) = \lvert\text{Hom}(a, b)\rvert$
Inverse of ζ	fine Möbius function μ	coarse Möbius function μ
Have M. inv. for:		

Fine and coarse: summary

A: suitably finite category. k: ring.

	Fine	Coarse
Incidence algebra	$k\mathbf{A} = \{\text{fns } \mathrm{arr}(\mathbf{A}) \to k\}$	$k_c\mathbf{A} = \{\text{fns } \mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k\}$
Multiplication	$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\beta(h)$	$(\alpha * \beta)(a, b) = \sum_b \alpha(a, b)\beta(b, c)$
Zeta function	$\zeta(f) \equiv 1$	$\zeta(a, b) = \lvert\mathrm{Hom}(a, b)\rvert$
Inverse of ζ	fine Möbius function μ	coarse Möbius function μ
Have M. inv. for:		
posets?	✓	✓

Fine and coarse: summary

A: suitably finite category. k: ring.

	Fine	Coarse
Incidence algebra	$k\mathbf{A} = \{\text{fns } \mathrm{arr}(\mathbf{A}) \to k\}$	$k_c\mathbf{A} = \{\text{fns } \mathrm{ob}(\mathbf{A}) \times \mathrm{ob}(\mathbf{A}) \to k\}$
Multiplication	$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\beta(h)$	$(\alpha * \beta)(a, b) = \sum_{b} \alpha(a, b)\beta(b, c)$
Zeta function	$\zeta(f) \equiv 1$	$\zeta(a, b) = \lvert\mathrm{Hom}(a, b)\rvert$
Inverse of ζ	fine Möbius function μ	coarse Möbius function μ
Have M. inv. for:		
posets?	✓	✓
monoids?	x	✓

Fine and coarse: summary

A: suitably finite category. k: ring.

	Fine	Coarse
Incidence algebra	$k\mathbf{A} = \{\text{fns arr}(\mathbf{A}) \to k\}$	$k_c\mathbf{A} = \{\text{fns ob}(\mathbf{A}) \times \text{ob}(\mathbf{A}) \to k\}$
Multiplication	$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\beta(h)$	$(\alpha * \beta)(a, b) = \sum_b \alpha(a, b)\beta(b, c)$
Zeta function	$\zeta(f) \equiv 1$	$\zeta(a, b) = \lvert\text{Hom}(a, b)\rvert$
Inverse of ζ	fine Möbius function μ	coarse Möbius function μ
Have M. inv. for:		
posets?	✓	✓
monoids?	x	✓
groupoids?	x	✓

2. Fine vs. coarse Möbius inversion

Overview

Number-theoretic Möbius inversion
(Möbius 1832)

Möbius inversion for posets
(Rota 1964, et al.)

Fine M. inv. for categories
(Leroux et al. 1975, 1980;
Haigh 1980)

Coarse M. inv. for categories
(Haigh 1980;
Leinster 2008)

Overview

Number-theoretic Möbius inversion
(Möbius 1832)

↓

Möbius inversion for posets
(Rota 1964, et al.)

↙ ↘

Fine M. inv. for categories
(Leroux et al. 1975, 1980;
Haigh 1980)

Coarse M. inv. for categories
(Haigh 1980;
Leinster 2008)

Functoriality

Let $F : \mathbf{A} \to \mathbf{B}$ be a suitably finite functor.

Functoriality

Let $F : \mathbf{A} \to \mathbf{B}$ be a suitably finite functor.

There is an induced linear map

$$\begin{array}{cccc} F_! : & k\mathbf{A} & \to & k\mathbf{B} \\ & \alpha & \mapsto & F_!\alpha \end{array}$$

Functoriality

Let $F : \mathbf{A} \to \mathbf{B}$ be a suitably finite functor.

There is an induced linear map

$$\begin{array}{rccc} F_! : & k\mathbf{A} & \to & k\mathbf{B} \\ & \alpha & \mapsto & F_!\alpha \end{array}$$

defined by

$$(F_!\alpha)(g) = \sum_{f \,:\, F(f)=g} \alpha(f)$$

($g \in \mathrm{arr}(\mathbf{B})$).

Functoriality

Let $F : \mathbf{A} \to \mathbf{B}$ be a suitably finite functor.

There is an induced linear map

$$F_! : \quad k\mathbf{A} \to k\mathbf{B}, \qquad \alpha \mapsto F_!\alpha$$

defined by

$$(F_!\alpha)(g) = \sum_{f \,:\, F(f)=g} \alpha(f)$$

($g \in \mathrm{arr}(\mathbf{B})$).

Proposition

$F_!$ is an algebra homomorphism for all k $\iff$

Functoriality

Let $F : \mathbf{A} \to \mathbf{B}$ be a suitably finite functor.

There is an induced linear map

$$\begin{array}{rccc} F_! : & k\mathbf{A} & \to & k\mathbf{B} \\ & \alpha & \mapsto & F_!\alpha \end{array}$$

defined by

$$(F_!\alpha)(g) = \sum_{f \,:\, F(f)=g} \alpha(f)$$

$(g \in \mathrm{arr}(\mathbf{B}))$.

Proposition

$F_!$ is an algebra homomorphism for all k $\iff$ F is bijective on objects.

Comparison theorem

Comparison theorem

Let **A** be a suitably finite category. Let k be a ring.

Comparison theorem

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

Write $C\mathbf{A}$ for the codiscrete category with the same objects as $\mathbf{A}$.

Comparison theorem

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

Write $C\mathbf{A}$ for the codiscrete category with the same objects as $\mathbf{A}$. Then:

- $k(C\mathbf{A}) = k_{\mathsf{c}}\mathbf{A}$

Comparison theorem

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

Write $C\mathbf{A}$ for the codiscrete category with the same objects as $\mathbf{A}$. Then:

- $k(C\mathbf{A}) = k_{\mathrm{c}}\mathbf{A}$
- the identity-on-objects functor $F : \mathbf{A} \to C\mathbf{A}$ induces a homomorphism

$$\Sigma = F_! \colon k\mathbf{A} \to k_{\mathrm{c}}\mathbf{A}$$

Comparison theorem

Let **A** be a suitably finite category. Let k be a ring.

Write $C\mathbf{A}$ for the codiscrete category with the same objects as **A**. Then:

- $k(C\mathbf{A}) = k_c\mathbf{A}$
- the identity-on-objects functor $F : \mathbf{A} \to C\mathbf{A}$ induces a homomorphism

  $$\Sigma = F_! \colon k\mathbf{A} \to k_c\mathbf{A},$$

  given by

  $$(\Sigma\alpha)(a, b) = \sum_{f \colon a \to b} \alpha(f).$$

Comparison theorem

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

Write $C\mathbf{A}$ for the codiscrete category with the same objects as $\mathbf{A}$. Then:

- $k(C\mathbf{A}) = k_c\mathbf{A}$
- the identity-on-objects functor $F : \mathbf{A} \to C\mathbf{A}$ induces a homomorphism

$$\Sigma = F_! \colon k\mathbf{A} \to k_c\mathbf{A},$$

given by

$$(\Sigma\alpha)(a, b) = \sum_{f \colon a \to b} \alpha(f).$$

Proposition (Haigh)

If $\mathbf{A}$ has fine Möbius inversion then $\mathbf{A}$ has coarse Möbius inversion

Comparison theorem

Let $\mathbf{A}$ be a suitably finite category. Let k be a ring.

Write $C\mathbf{A}$ for the codiscrete category with the same objects as $\mathbf{A}$. Then:

- $k(C\mathbf{A}) = k_c\mathbf{A}$
- the identity-on-objects functor $F : \mathbf{A} \to C\mathbf{A}$ induces a homomorphism

$$\Sigma = F_! \colon k\mathbf{A} \to k_c\mathbf{A},$$

  given by

$$(\Sigma\alpha)(a, b) = \sum_{f\colon a \to b} \alpha(f).$$

Proposition (Haigh)

If $\mathbf{A}$ *has fine Möbius inversion then* $\mathbf{A}$ *has coarse Möbius inversion, with*

$$\mu_{coarse}(a, b) = \sum_{f\colon a \to b} \mu_{fine}(f).$$

Comparison theorem

Let **A** be a suitably finite category. Let k be a ring.

Write $C\mathbf{A}$ for the codiscrete category with the same objects as **A**. Then:

- $k(C\mathbf{A}) = k_c\mathbf{A}$
- the identity-on-objects functor $F : \mathbf{A} \to C\mathbf{A}$ induces a homomorphism

  $$\Sigma = F_! \colon k\mathbf{A} \to k_c\mathbf{A},$$

  given by

  $$(\Sigma\alpha)(a, b) = \sum_{f \colon a \to b} \alpha(f).$$

Proposition (Haigh)

*If **A** has fine Möbius inversion then **A** has coarse Möbius inversion, with*

$$\mu_{coarse}(a, b) = \sum_{f \colon a \to b} \mu_{fine}(f).$$

Proof $\zeta_{\text{coarse}} = \Sigma\zeta_{\text{fine}}$, so $\mu_{\text{coarse}} = \Sigma\mu_{\text{fine}}$.

3. How important is composition in a category?

How important is composition for fine Möbius inversion?

How important is composition for fine Möbius inversion?

Proposition (Haigh) *If* **A** *has fine M. inv. then* **A** *has coarse M. inv., with*

$$\mu_{coarse}(a, b) = \sum_{f\colon a \to b} \mu_{fine}(f).$$

How important is composition for fine Möbius inversion?

Proposition (Haigh) *If* $\mathbf{A}$ *has fine M. inv. then* $\mathbf{A}$ *has coarse M. inv., with*

$$\mu_{coarse}(a, b) = \sum_{f\colon a \to b} \mu_{fine}(f).$$

Corollary (Menni) *If* $\mathbf{A}$ *has fine Möbius inversion then, for all* $a, b \in \mathbf{A}$,

$$\sum_{f\colon a \to b} \mu(f)$$

is independent of the composition and identities in $\mathbf{A}$.

How important is composition for fine Möbius inversion?

Proposition (Haigh) *If* **A** *has fine M. inv. then* **A** *has coarse M. inv., with*

$$\mu_{coarse}(a, b) = \sum_{f\colon a \to b} \mu_{fine}(f).$$

Corollary (Menni) *If* **A** *has fine Möbius inversion then, for all* $a, b \in \mathbf{A}$,

$$\sum_{f\colon a \to b} \mu(f)$$

is independent of the composition and identities in **A**.

Corollary *If* **A** *has fine Möbius inversion then*

$$\sum_{f \in \mathrm{arr}(\mathbf{A})} \mu(f)$$

is independent of the composition and identities in **A**.

Euler characteristic

Euler characteristic

Let **A** be a suitably finite category with coarse Möbius inversion.

Euler characteristic

Let **A** be a suitably finite category with coarse Möbius inversion.

The Euler characteristic of **A** is

$$\chi(\mathbf{A}) = \sum_{a,b} \mu(a, b).$$

Euler characteristic

Let **A** be a suitably finite category with coarse Möbius inversion.

The Euler characteristic of **A** is

$$\chi(\mathbf{A}) = \sum_{a,b} \mu(a, b).$$

Theorem $\chi(\mathbf{A}) = \chi(B\mathbf{A})$.

Euler characteristic

Let **A** be a suitably finite category with coarse Möbius inversion.

The Euler characteristic of **A** is

$$\chi(\mathbf{A}) = \sum_{a,b} \mu(a, b).$$

Theorem $\chi(\mathbf{A}) = \chi(B\mathbf{A})$.

Corollary $\chi(B\mathbf{A})$ *is independent of the composition and identities in* **A**.

Euler characteristic

Let **A** be a suitably finite category with coarse Möbius inversion.

The Euler characteristic of **A** is

$$\chi(\mathbf{A}) = \sum_{a,b} \mu(a, b).$$

Theorem $\chi(\mathbf{A}) = \chi(B\mathbf{A})$.

Corollary $\chi(B\mathbf{A})$ *is independent of the composition and identities in* **A**.

Corollary *If* **A** *has fine Möbius inversion then*

$$\sum_{f \in \mathrm{arr}(\mathbf{A})} \mu(f) = \chi(B\mathbf{A}).$$

4. Postscript:

A theorem of Beck–Chevalley type

Functoriality revisited

Functoriality revisited

Recall: A bijective-on-objects functor $F : \mathbf{A} \to \mathbf{B}$ induces a homomorphism

$$k\mathbf{A} \xrightarrow{F_!} k\mathbf{B}$$

where $(F_! \alpha)(g) = \sum_{f \,:\, Ff = g} \alpha(f)$.

Functoriality revisited

Recall: A bijective-on-objects functor $F : \mathbf{A} \to \mathbf{B}$ induces a homomorphism

$$k\mathbf{A} \xrightarrow{F_!} k\mathbf{B}$$

where $(F_! \alpha)(g) = \sum_{f \,:\, Ff = g} \alpha(f)$.

Also: A functor $F : \mathbf{A} \to \mathbf{B}$ induces a linear map

$$k\mathbf{A} \xleftarrow{F^*} k\mathbf{B}$$

where $(F^* \beta)(f) = \beta(Ff)$.

Functoriality revisited

Recall: A bijective-on-objects functor $F : \mathbf{A} \to \mathbf{B}$ induces a homomorphism

$$k\mathbf{A} \xrightarrow{F_!} k\mathbf{B}$$

where $(F_!\alpha)(g) = \sum_{f\,:\,Ff=g} \alpha(f)$.

Also: A functor $F : \mathbf{A} \to \mathbf{B}$ induces a linear map

$$k\mathbf{A} \xleftarrow{F^*} k\mathbf{B}$$

where $(F^*\beta)(f) = \beta(Ff)$. It is an algebra homomorphism if F has unique lifting of factorizations.

Functoriality revisited

Recall: A bijective-on-objects functor $F : \mathbf{A} \to \mathbf{B}$ induces a homomorphism

$$k\mathbf{A} \xrightarrow{F_!} k\mathbf{B}$$

where $(F_!\alpha)(g) = \sum_{f\,:\,Ff=g} \alpha(f)$.

Also: A functor $F : \mathbf{A} \to \mathbf{B}$ induces a linear map

$$k\mathbf{A} \xleftarrow{F^*} k\mathbf{B}$$

where $(F^*\beta)(f) = \beta(Ff)$. It is an algebra homomorphism if F has unique lifting of factorizations.

Question: How are these covariant and contravariant processes related?

A theorem of Beck–Chevalley type

A theorem of Beck–Chevalley type

Proposition *Let*

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{F'} & \mathbf{B} \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} \end{array}$$

be a pullback square of suitably finite categories,
such that F is bijective on objects and G has unique lifting of factorizations.

A theorem of Beck–Chevalley type

Proposition *Let*

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{F'} & \mathbf{B} \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} \end{array}$$

be a pullback square of suitably finite categories,
such that F is bijective on objects and G has unique lifting of factorizations.

Then F' is bijective on objects, G' has unique lifting of factorizations,

A theorem of Beck–Chevalley type

Proposition *Let*

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{F'} & \mathbf{B} \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} \end{array}$$

be a pullback square of suitably finite categories,
such that F is bijective on objects and G has unique lifting of factorizations.

Then F' is bijective on objects, G' has unique lifting of factorizations,
and the square

$$\begin{array}{ccc} k\mathbf{D} & \xrightarrow{F'_!} & k\mathbf{B} \\ {\scriptstyle G'^*}\uparrow & & \uparrow{\scriptstyle G^*} \\ k\mathbf{A} & \xrightarrow[F_!]{} & k\mathbf{C} \end{array}$$

commutes.

Summary

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category
    - does not exist for many categories

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category
    - does not exist for many categories
  - Coarse Möbius inversion:

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category
    - does not exist for many categories
  - Coarse Möbius inversion:
    - does not depend on the composition in the category

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category
    - does not exist for many categories
  - Coarse Möbius inversion:
    - does not depend on the composition in the category
    - exists for most categories

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category
    - does not exist for many categories
  - Coarse Möbius inversion:
    - does not depend on the composition in the category
    - exists for most categories
- Fine M. inversion $\Rightarrow$ coarse M. inversion, with $\mu(a, b) = \sum_{f\colon a \to b} \mu(f)$

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category
    - does not exist for many categories
  - Coarse Möbius inversion:
    - does not depend on the composition in the category
    - exists for most categories
- Fine M. inversion $\Rightarrow$ coarse M. inversion, with $\mu(a, b) = \sum_{f\colon a \to b} \mu(f)$
- The Euler characteristic of a category $\mathbf{A}$:

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category
    - does not exist for many categories
  - Coarse Möbius inversion:
    - does not depend on the composition in the category
    - exists for most categories
- Fine M. inversion $\Rightarrow$ coarse M. inversion, with $\mu(a, b) = \sum_{f\colon a \to b} \mu(f)$
- The Euler characteristic of a category $\mathbf{A}$:
  - is $\chi(B\mathbf{A}) = \sum_{a,b} \mu(a, b) = \sum_{f} \mu(f)$

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
  - Fine Möbius inversion:
    - depends on the composition in the category
    - does not exist for many categories
  - Coarse Möbius inversion:
    - does not depend on the composition in the category
    - exists for most categories
- Fine M. inversion $\Rightarrow$ coarse M. inversion, with $\mu(a, b) = \sum_{f\colon a \to b} \mu(f)$
- The Euler characteristic of a category $\mathbf{A}$:
  - is $\chi(B\mathbf{A}) = \sum_{a,b} \mu(a, b) = \sum_{f} \mu(f)$
  - does not depend on the composition in $\mathbf{A}$.

Summary

- Number-theoretic M. inversion is generalized by M. inversion for posets
- M. inversion for posets can be generalized to categories in two ways:
    - Fine Möbius inversion:
        - depends on the composition in the category
        - does not exist for many categories
    - Coarse Möbius inversion:
        - does not depend on the composition in the category
        - exists for most categories
- Fine M. inversion ⇒ coarse M. inversion, with $\mu(a, b) = \sum_{f\colon a \to b} \mu(f)$
- The Euler characteristic of a category **A**:
    - is $\chi(B\mathbf{A}) = \sum_{a,b} \mu(a, b) = \sum_{f} \mu(f)$
    - does not depend on the composition in **A**.
- Throwing away the composition of a category might seem extravagant, but it's surprising how much remains.

References

- Mireille Content, François Lemay, Pierre Leroux, Catégories de Möbius et fonctorialités: un cadre général pour l'inversion de Möbius. *Journal of Combinatorial Theory, Series A* 28 (1980), 169–190.
- John Haigh, On the Möbius algebra and the Grothendieck ring of a finite category. *Journal of the London Mathematical Society (2)* 21 (1980), 81–92.
- Tom Leinster, The Euler characteristic of a category. *Documenta Mathematica* 13 (2008), 21–49.
- Pierre Leroux, Les catégories de Möbius. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 16 (1975), 280–282.
- Gian-Carlo Rota, On the foundations of combinatorial theory I: theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 2 (1964), 340–368.