INTRODUCTION TO HIGHER (ESPECIALLY GLOBULAR) OPERADS

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Reading

Higher-dimensional theories:

Albert Burroni, 1971,

"T-catégories (catégories dans un triple)"

Globular operads, and application to n-categories:

Michael Batanin, 1998,

"Monoidal globular categories as a natural environment for the theory of weak n-categories"

Discussion of relationship between these two:

Tom Leinster, 2004,

"Higher Operads, Higher Categories".

Plan

I. Universal algebra

(or: some lower-dimensional category theory)

II. Globular operads

(or: some higher-dimensional category theory)

Executive Summary

In ordinary, set-based, algebra,

an operation takes an n-tuple of elements

as input, e.g.

$$(x_1, x_2, x_3) \longmapsto x_1 \cdot (x_2 \cdot x_3).$$

The collection of all operations forms

an operad (or something like it).

In higher-dimensional algebra,

an operation may take a diagram of data

as input, e.g.

$$\begin{tikzcd}[column sep=large]
x \arrow[r, bend left=70, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=70, "f_3"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\alpha_1"] \arrow[r, phantom, shift right=4ex, "\Downarrow\alpha_2"] & y \arrow[r, bend left=70, "g_1"] \arrow[r, "g_2"] \arrow[r, bend right=70, "g_3"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\beta_1"] \arrow[r, phantom, shift right=4ex, "\Downarrow\beta_2"] & z
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}[column sep=huge]
x \arrow[r, bend left=60, "g_1 \circ f_1"] \arrow[r, bend right=60, "g_3 \circ f_3"'] \arrow[r, phantom, "\Downarrow (\beta_2\beta_1) * (\alpha_2\alpha_1)"] & z
\end{tikzcd}.$$

The collection of all operations forms

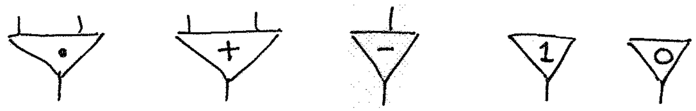
a "higher operad" (in this picture, globular).

I. UNIVERSAL ALGEBRA

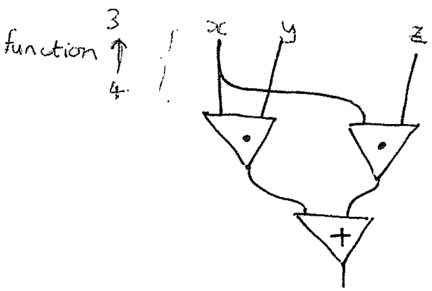
What's an "algebraic theory"?

The theory of rings should be a typical example.

Have basic operations



and from these, can build more: e.g.



"$(x,y,z) \mapsto x.y + x.z$"

A <u>(finitary algebraic) theory</u> consists of

- a sequence $P_0, P_1, \ldots$ of sets
  (think of P_n as the set of operations with n inputs; here, $P_n = \mathbb{Z}\langle t_1, \ldots, t_n\rangle$)
- maps $P_k \times P_{n_1} \times \cdots \times P_{n_k} \longrightarrow P_{n_1 + \cdots + n_k}$
- an element $id_P \in P_1$
- maps $\mathrm{Set}(m,n) \times P_m \longrightarrow P_n$

satisfying predictable axioms.

Models / algebras for a theory

A ring is a set X together with, for each $\theta \in P_n = \mathbb{Z}\langle t_1, \ldots, t_n\rangle$, a map $\bar{\theta}: X^n \to X$, satisfying axioms.

In general, a <u>model</u> or <u>algebra</u> for a theory P is a set X together with, for each $\theta \in P_n$, a map $\bar{\theta}: X^n \to X$, satisfying axioms.

In evident way, there's a functor

$$\begin{array}{ccc} (\text{Theories})^{op} & & P \\ \downarrow{\scriptstyle \underline{\mathrm{Alg}}} & & \downarrow \\ \mathrm{CAT} & & \underline{\mathrm{Alg}}(P) = (\text{algebras for } P). \end{array}$$

Theories vs. monads

Let P be a theory. There's an adjunction

$$\begin{tikzcd} \mathrm{Alg}(P) \arrow[d, shift left=1.5ex, "\text{Underlying}"] \\ \mathbf{Set}, \arrow[u, shift left=1.5ex, "\text{Free}"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

inducing a monad $T_P = U \circ F$ on **Set**.

(In fact, $T_P(X) = \left(\coprod_n P_n \times X^n \right)/\sim$.)

Pursuing this, get equivalence

$$\begin{tikzcd} P \arrow[rr, mapsto] & & T_P \\ (\text{Theories})^{op} \arrow[rr, phantom, "\simeq"] \arrow[dr, "\mathrm{Alg}"'] & & (\text{Finitary monads on } \mathbf{Set})^{op} \arrow[dl, "\mathrm{Alg}"] \\ & \mathbf{CAT} & \end{tikzcd}$$

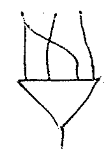
$\mathrm{Alg}(P) \simeq \mathrm{Alg}(T_P)$

between two different ways of formalizing "algebraic theory".

How operads fit in

The definition of "theory" included actions

$$\{\text{functions } m \to n\} \times P_m \to P_n .$$



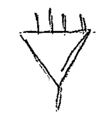
If we change this to

$$\{\text{bijections } m \to n\} \times P_m \to P_n$$

then we get the definition of symmetric operad.

If we change it to

$$\text{“}\{\text{equalities } m \to n\} \times P_m \to P_n\text{”}$$



— that is, drop it altogether — then we get the definition of planar (non-symmetric) operad.

For the rest of Part I, "operad" means "planar operad".

A definition of operad

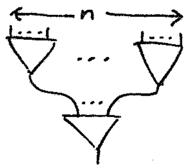
Key point: $\mathbf{Set}/\mathbb{N}$ is a monoidal category in an interesting way.

An object $\begin{matrix} Q \\ \downarrow \\ \mathbb{N} \end{matrix}$ of $\mathbf{Set}/\mathbb{N}$ is a family $(Q_n)_{n\in\mathbb{N}}$ of sets

Given $P, Q \in \mathbf{Set}/\mathbb{N}$, define $Q\otimes P \in \mathbf{Set}/\mathbb{N}$ by

$$(Q\otimes P)_n = \coprod_{n_1+\cdots+n_k=n} Q_k \times P_{n_1} \times \cdots \times P_{n_k}$$

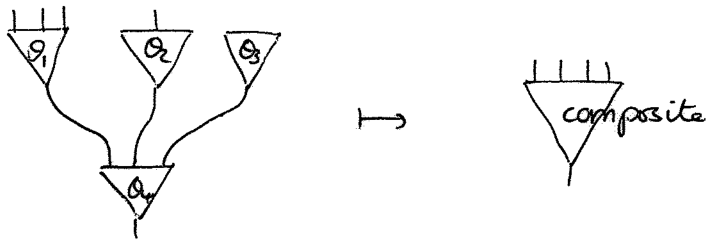
$= \{$ diagrams



$\}$

Defn: An <u>operad</u> is a monoid in $(\mathbf{Set}/\mathbb{N}, \otimes)$.

Example of composition in an operad:



Operads vs. monads

Let P be an operad. Then there's an adjunction

$$\begin{tikzcd} \mathrm{Alg}(P) \arrow[d, shift left, "\text{Underlying}"] \\ \mathbf{Set} \arrow[u, shift left, "\text{Free}"] \end{tikzcd}$$

inducing a monad $T_P = U \circ F$ on $\mathbf{Set}$. In fact,

$$T_P(X) = \coprod_{n\in\mathbb{N}} P_n \times X^n$$

$(X \in \mathbf{Set})$.

The diagram

$$\begin{tikzcd} (\text{Operads})^{op} \arrow[rr, "\exists!"] \arrow[dr, "\mathrm{Alg}"'] & & (\text{Finitary monads on Set})^{op} \arrow[dl, "\mathrm{Alg}"] \\ & \mathbf{CAT} & \end{tikzcd}$$

$$P \longmapsto T_P$$

commutes, i.e. $\mathrm{Alg}(P) = \mathrm{Alg}(T_P)$.

E.g.: Let $P = 1$, i.e. $P_n = 1 = \{*\}$ for all n.

Then $T_P(X) = \coprod_n X^n$ and $\mathrm{Alg}(P) = \mathrm{Alg}(T_P)$

$=$ <u>Monoid</u>.

Operads vs. monads (more subtle...)

Write $\underline{S}$ for the free monoid monad on Set.

For any operad P, have map

$$T_P = \coprod_n P_n \times (-)^n \xrightarrow{\text{proj}_P} \coprod_n (-)^n = S.$$

In fact,

- the natural transformation proj_P is <u>cartesian</u> (its naturality squares are pullbacks)
- the monad T_P is <u>cartesian</u> (its functor part preserves pullbacks, and its natural transformation parts are cartesian)
- proj_P commutes with the monad structures,

and even better,

- an operad can be <u>defined</u> as a cartesian monad on Set equipped with a cartesian monad map to S.

Warning! Non-isomorphic operads can have the same underlying monad. So <u>need</u> the proj_P's!

Executive summary, revisited

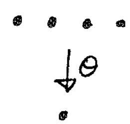
Operations in an ordinary operad take a <u>sequence</u> of inputs.

Operations in a higher operad will take a <u>higher-dimensional diagram</u> of inputs.

Burroni's idea:

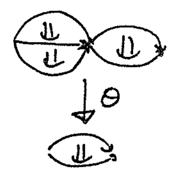
- the concept of "sequence" is contained in the free monoid monad S
- can try replacing S by a different monad.

Ordinary operads have something to do with diagrams like

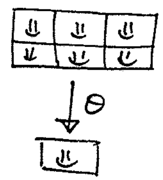


$(\in P_4)$

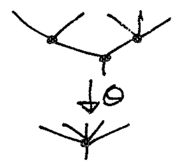
Globular operads " " " "



Cubical operads " " " "



Opetopic operads " " " "



Executive summary, revisited

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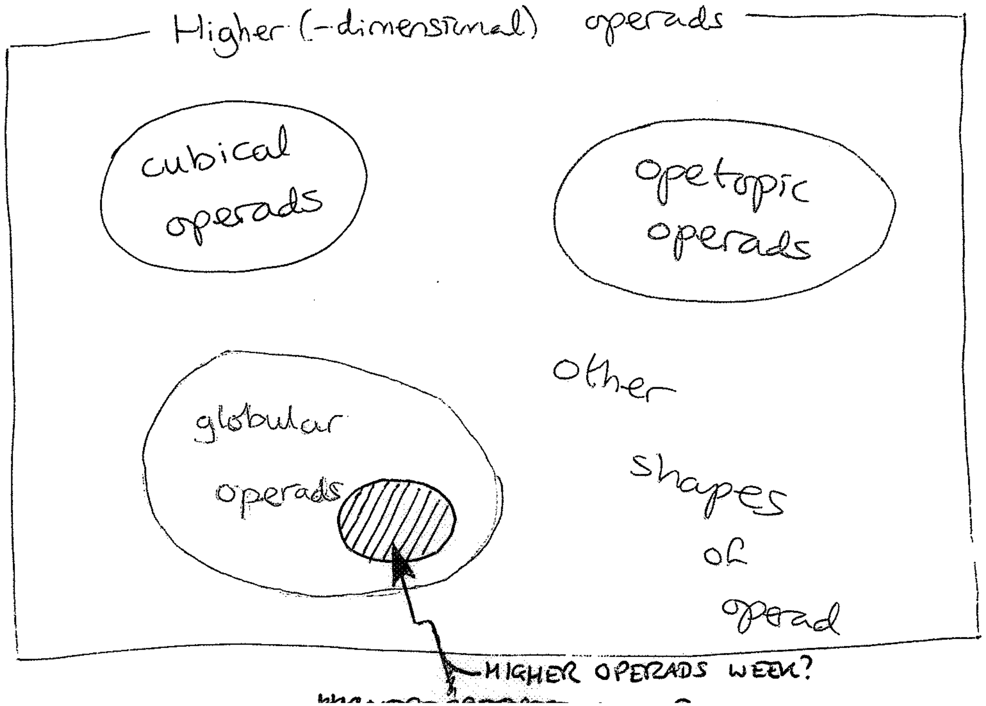


II. GLOBULAR OPERADS

Globular sets

Let $n \in \mathbb{N} \cup \{\infty\}$.

An n-globular set (or n-graph), X, consists of sets and functions

$$X_n \overset{s}{\underset{t}{\rightrightarrows}} \cdots\cdots \overset{s}{\underset{t}{\rightrightarrows}} X_1 \overset{s}{\underset{t}{\rightrightarrows}} X_0$$

such that for $\alpha \in X_d$ $(d \geq 2)$,

$$ss(\alpha) = st(\alpha) \quad \text{and} \quad ts(\alpha) = tt(\alpha).$$

Think of X_d as the set of d-dimensional cells, s as source/domain, & t as target/codomain:

$$\overset{x}{\bullet} \in X_0, \qquad \underset{\substack{x\\=s(f)}}{\bullet} \xrightarrow{f} \underset{\substack{y\\=t(f)}}{\bullet} \in X_1, \qquad \begin{tikzcd} x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"] & y \end{tikzcd} \in X_2$$

(Here the equations say that $s(f)=s(g)$ & $t(f)=t(g)$: "globularity".)

A globular set is an ∞-globular set.

We focus on $n = \infty$.

Globular sets and strict ∞-categories

There is an adjunction

$$\begin{array}{c} \mathrm{Strict}\,\infty\mathrm{Cat} \\ \mathrm{Free} \uparrow \dashv \downarrow \mathrm{Underlying} \\ \mathrm{GlobSet} \end{array}$$

inducing a monad $S = UF$ on GlobSet.

What's $S(1)$?

"1" here is the terminal globular set, with just one cell in each dimension. So it's (the underlying globular set of) the free strict ∞-category on this.

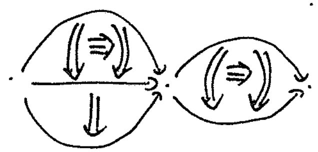
It's the analogue of $\mathbb{N}$.

But what does it look like?

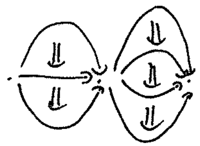
Descriptions of $S(1)$

(it's a globular set!)

- An element of $S(1)_d$ is a <u>globular pasting diagram</u> of dimension $\leq d$ — that is, a shape that in a strict ∞-category can be composed to make a d-cell. E.g.

 $\in S(1)_3$.

- Recursively, $S(1)_0 = \{\bullet\}$ and $S(1)_{d+1}$ = free monoid on $S(1)_d$

E.g.  $\in S(1)_2$ corresponds to

$(\underbrace{\bullet \to \bullet \to \bullet}_{\in S(1)_1}, \underbrace{\bullet \to \bullet \to \bullet \to \bullet}_{\in S(1)_1})$, which in turn corresponds to

$((\underbrace{\boxed{\bullet}}_{\in S(1)_0}, \bullet), (\bullet, \bullet, \bullet))$.

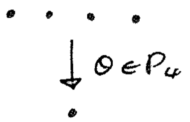
- $S(1)_d = \{$trees of height $\leq d\}$, e.g. this is  $\in S(1)_2$

- $S(1)_d = \{$diagrams $r_d \to r_{d-1} \to \cdots \to r_1 \to 1$ in category of finite ordinals$\}$

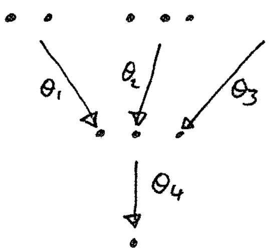
Informal "definition" of globular operad

An ordinary operad consists of ...

- for each $n \in \mathbb{N}$, a set P_n of "n-ary operations":



- composition: e.g. operations



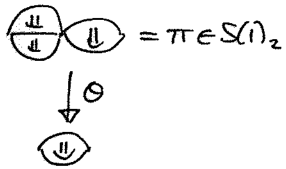
compose to give an operation

- an identity,

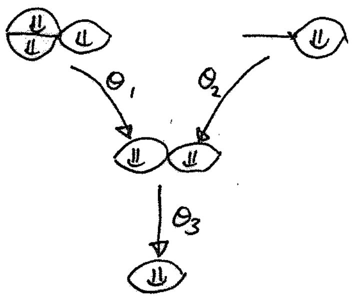
satisfying associativity & unit laws.

A globular operad consists of ...

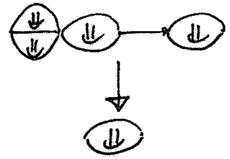
- for each $d \geq 0$ and $\pi \in S(1)_d$, a set P_π of "π-ary operations":



- composition: e.g. operations



compose to give an operation



- identities,

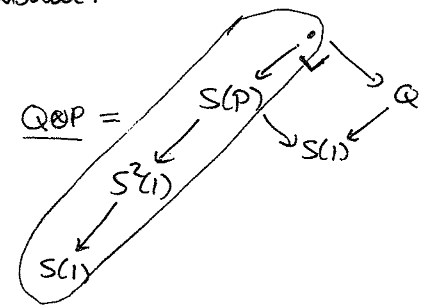
satisfying associativity & unit laws.

Formal definition of globular operad

Key point: $\mathbf{GlobSet}/S(1)$ is a monoidal category in an interesting way.

Given $\begin{array}{c} Q \\ \downarrow \\ S(1) \end{array}$, $\begin{array}{c} P \\ \downarrow \\ S(1) \end{array}$ $\in \mathbf{GlobSet}/S(1)$, define $\underline{Q \otimes P}$ by pullback:

$\underline{Q\otimes P}$ =



(The $\otimes$ on $\mathbf{Set}/\mathbb{N}$ can be described in the same way.)

Defn: A $\underline{\text{globular operad}}$ is a monoid in $(\mathbf{GlobSet}/S(1), \otimes)$.

Any globular operad has an underlying $\underline{\text{collection}}$, i.e. an object of $\mathbf{GlobSet}/S(1)$.

It's a family $(P_\pi)_{d \geq 0,\, \pi \in S(1)_d}$ of sets,

with sources & target maps, e.g.

$$P_{\Downarrow\Downarrow} \rightrightarrows P_{\cdot \to \cdot \to \cdot}.$$

"Globular tuples"

Given a set X and $n \in \mathbb{N}$, get set X^n of n-tuples in X.

Given a glob. set X and $\pi \in S(1)_d$ (some d), get set X^π of "π-tuples in X".

E.g.: if $\pi = $
$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \end{tikzcd}$$
$\in S(1)_2$ then

$$X^\pi = \left\{ \text{diagrams } \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, "h"] & z \end{tikzcd} \text{ in } X \right\}$$
$$= \{ (x,y,z,f,g,h,\alpha) \in X_0^3 \times X_1^3 \times X_2 \mid s(f) = x,\ t(\alpha) = g, \ldots \}.$$

(Can define formally via monoidal structure of Globset$/S(1)$.)

We have
$$\boxed{S(X)_d = \coprod_{\pi \in S(1)_d} X^\pi}.$$

Algebras for globular operads

Let P be a globular operad.

There's an induced monad $\underline{T_P}$ on Globset: $\forall X \in$ Globset, $d \geq 0$,
$$(T_P(X))_d = \coprod_{\pi \in S(1)_d} P_\pi \times X^\pi.$$

Defn: A <u>P-algebra</u> is an algebra for T_P.

So, a P-algebra consists of:

- a globular set X
- for each operation $\theta \in P_\pi$ ($\pi \in S(1)_d$), a function $\bar\theta : X^\pi \to X_d$

satisfying axioms.

E.g.: We've been drawing $\theta \in P_{\pi}$ (for $\pi = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \end{tikzcd}$) as
$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \\ & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[from=1-2, to=2-2, phantom, "\downarrow\theta"] \end{tikzcd}$$

If X is a P-algebra, we get an actual function
$$\left\{ \text{diagrams } \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, "h"] & z \end{tikzcd} \text{ in } X \right\}$$
$$\downarrow \bar\theta$$
$$\left\{ \text{diagrams } \begin{tikzcd} v \arrow[r, bend left, "j"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & w \end{tikzcd} \text{ in } X \right\} = X_2.$$

Examples of algebras for globular operads

- The terminal globular operad 1 has $1_\pi = 1$ for all π. It's easy to see that $T_1 \cong S$, so a 1-algebra is just a strict ∞-category: one operation of each globular arity.

- One strategy for giving a definition of weak ∞-category: choose a suitable globular operad P and declare a weak ∞-category to be a P-algebra.

  Then want lots of operations of each arity π. E.g. want at least 2 elements of $P_{\bullet\to\bullet\to\bullet\to\bullet}$ because in a weak ∞-category we want $(h\circ g)\circ f \neq h\circ(g\circ f)$.

Globular operads vs. monads

For any globular operad P, have projection map

$$T_P \xrightarrow{\mathrm{proj}_P} S$$

given by

$$(T_P(X))_d = \coprod_{\pi\in S(1)_d} P_\pi \times X^\pi \xrightarrow{\mathrm{proj}} \coprod_{\pi\in S(1)_d} X^\pi = (S(X))_d.$$

Pursuing this thought, discover:

a globular operad can be defined as a cartesian monad on GlobSet equipped with a cartesian monad map to S

(just as for ordinary operads).

Question: Can non-isomorphic globular operads have the same underlying monad?

(Guess: no.)