INTRODUCTION TO HIGHER (ESPECIALLY GLOBULAR) OPERADS

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Reading

Higher-dimensional theories:
Albert Burroni, 1971,
"T-categories (catégories dans un triple)"

Globular operads, and application to \textit{n}-categories:
Michael Batanin, 1998,
"Monoidal globular categories as a natural environment for the theory of weak \textit{n}-categories"

Discussion of relationship between these two:
Tom Leinster, 2004,
"Higher Operads, Higher Categories".
Executive Summary

In ordinary, set-based, algebra, an operation takes an $n$-tuple of elements as input, e.g.

$$(x_1, x_2, x_3) \mapsto x_1 \cdot (x_2 \cdot x_3).$$

The collection of all operations forms an operad (or something like it).

In higher-dimensional algebra, an operation may take a diagram of data as input, e.g.

The collection of all operations forms a "higher operad" (in this picture, globular).
I. UNIVERSAL ALGEBRA

What's an "algebraic theory"?

The theory of rings should be a typical example.

Have basic operations

\[
\begin{array}{cccc}
\cdot & + & - & 0
\end{array}
\]

and from these, can build more: e.g.

\[ (x, y, z) \mapsto x \cdot y + x \cdot z \]

A (finitary algebraic) theory consists of

- a sequence \( P_0, P_1, \ldots \) of sets
  (think of \( P_n \) as the set of operations with \( n \) inputs; here, \( P_n = \mathbb{Z} < t_1, \ldots, t_n > \))
- maps \( P_n \times P_n \times \cdots \times P_n \rightarrow P_{n+m} \)
- an element \( \text{id}_P \in P_1 \)
- maps \( \text{Set}(m,n) \times P_m \rightarrow P_n \)

satisfying predictable axioms.

Models/algebras for a theory

A ring is a set \( X \) together with,
for each \( \Theta \in P_n = \mathbb{Z} < t_1, \ldots, t_n > \), a map \( \bar{\Theta} : X^n \rightarrow X \), satisfying axioms.

In general, a model or algebra for a theory \( P \) is a set \( X \) together with,
for each \( \Theta \in P_n \), a map \( \bar{\Theta} : X^n \rightarrow X \), satisfying axioms.

In evident way, there's a functor

\[
\begin{array}{ccc}
(\text{Theories})^\text{op} & \rightarrow & \text{P} \\
\downarrow \text{Alg} & & \downarrow \text{Alg}(P) \equiv \text{(algebras for } P\text{)} \\
\text{CAT} & & \text{CAT}
\end{array}
\]
Theories vs. monads

Let $P$ be a theory. There's an adjunction

$$\begin{align*}
\text{Alg}(P) & \xrightarrow{\text{Free}} \text{Underlying} \\
& \Downarrow \ \Uparrow \text{Set}
\end{align*}$$

inducing a monad $T_P = \text{U} \circ \text{F}$ on $\text{Set}$.

(In fact, $T_P(X) = (\coprod_n P_n \times X^n)/\sim$)

Pursuing this, get equivalence

$$
\begin{align*}
P^\text{op} & \cong (\text{Finitary monads on Set})^\text{op} \\
& \xrightarrow{\text{Alg}} (\text{Theories})^\text{op} \\
& \xrightarrow{\text{Alg}} \text{CAT}
\end{align*}
$$

between two different ways of formalizing

"algebraic theory".

How operads fit in

The definition of "theory" included actions

$$\{ \text{functions } m \to n \times P_m \to P_n \}$$

If we change this to

$$\{ \text{bijections } m \to n \times P_m \to P_n \}$$

then we get the definition of symmetric operad.

If we change it to

$$\{ \text{equalities } m \to n \times P_m \to P_n \}$$

—that is, drop it altogether—then we get the definition of planar (non-symmetric) operad.

For the rest of Part I, "operad" means "planar operad".
A definition of operad

Key point: $\text{Set}_{/N}$ is a monoidal category in an interesting way.

An object $Q$ of $\text{Set}_{/N}$ is a family $(q_n)_n$ of $\text{Set}$.

Given $P,Q \in \text{Set}_{/N}$, define $Q \otimes P \in \text{Set}_{/N}$ by

$$(Q \otimes P)_n = \coprod_{n_1 + \cdots + n_k = n} q_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}$$

$$= \{ \text{diagrams } \begin{array}{ccc} \cdots & \vdots & \vdots \\ & \uparrow & \uparrow \\ & \uparrow & \uparrow \\ & \uparrow & \uparrow \\ & \uparrow & \uparrow \\ & \uparrow & \uparrow \\ & \uparrow & \uparrow \\ & \uparrow & \uparrow \\ & \uparrow & \uparrow \\ & \uparrow & \uparrow \\ \end{array} \} \in P \}$$

in $Q$.

Defn: An operad is a monoid in $(\text{Set}_{/N}, \otimes)$.

Example of composition in an operad:

$\begin{array}{cccccc}
\oplus & \oplus & \oplus & \oplus & \oplus \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & & & \text{composite} \\
\end{array}$

Operads vs. monads

Let $P$ be an operad. Then there's an adjunction

$$\begin{array}{ccc}
\text{Alg}(P) & \overset{\text{Free}}{\longrightarrow} & \text{Underlying} \\
\downarrow & & \downarrow \\
\text{Set} & & \text{Set} \\
\end{array}$$

inducing a monad $T_P = \text{UoF}$ on Set. In fact,

$$T_P(X) = \coprod_{n \in \mathbb{N}} P_n \times X^n$$

($X \in \text{Set}$).

The diagram

$$\text{(Operads)}^{op} \longrightarrow \text{Finitary monads on Set}$$

commutes, i.e., $\text{Alg}(P) = \text{Alg}(T_P)$.

E.g.: Let $P = 1$, i.e. $P_n = \{ * \}$ for all $n$.

Then $T_P(X) = \coprod_n X^n$ and $\text{Alg}(P) = \text{Alg}(T_P)$

$= \text{Monoid}$. 
Operads vs. monads (more subtle...)

Write $S$ for the free monoid monad on $\text{Set}$. For any operad $P$, have map

$$T_p = \bigsqcup_n P_n \times (-)^n \xrightarrow{\text{proj}} \bigsqcup_n (-)^n = S.$$ 

In fact,

- the natural transformation $\text{proj}$ is cartesian (its naturality squares are pullbacks)
- the monad $T_p$ is cartesian (its functor part preserves pullbacks, and its natural transformation parts are cartesian)
- $\text{proj}$ commutes with the monad structures,

and even better,

- an operad can be defined as a cartesian monad on $\text{Set}$ equipped with a cartesian monad map to $S$.

Warning! Non-isomorphic operads can have the same underlying monad. So need the proj's!

Executive summary, revisited

Operations in an ordinary operad take a sequence of inputs.

Operations in a higher operad will take a higher-dimensional diagram of inputs.

Burroni’s idea:

- the concept of “sequence” is contained in the free monoid monad $S$
- can try replacing $S$ by a different monad.

Ordinary operads have something to do with diagrams like

Globular operads

Cubical operads

Opetopic operads

These diagrams illustrate how the operad structure is defined using the monad $T_p$.
Higher operads

**cubical operads**

**opetopic operads**

**globular operads**

**other shapes of operad**
Globular sets and strict ∞-categories

There is an adjunction

\[ \text{Strict} \xrightarrow{\text{Free}} \text{Set} \]

\[ \Downarrow \text{Underlying} \]

\[ \text{GlobSet} \]

inducing a monad \( S = \text{UF} \) on \( \text{GlobSet} \).

What's \( S(1) \)?

"1" here is the terminal globular set, with just one cell in each dimension. So it's (the underlying globular set of) the free strict ∞-category on this.

It's the analogue of \( \mathbb{N} \).

But what does it look like?

Descriptions of \( S(1) \)

- An element of \( S(1)_d \) is a globular pasting
  diagram of dimension \( \leq d \) — that is, a shape that in a strict ∞-category can be composed to make a \( d \)-cell. E.g.

  \[
  \begin{array}{c}
  \text{arrow}
  \end{array}
  \]

  \( \in S(1)_3 \).

- Recursively, \( S(1)_0 = \{ \ast \} \) and \( S(1)_d = \text{free monoid on } S(1)_0 \)

  E.g.

  \[
  \begin{array}{c}
  \text{arrow}
  \end{array}
  \]

  \( \in S(1)_2 \) corresponds to

  \[
  (\ast \rightarrow \ast, \ast \rightarrow \ast, \ast \rightarrow \ast, \ast \rightarrow \ast)
  \]

  which in turn corresponds to

  \[
  (\ast, \ast), (\ast, \ast, \ast, \ast, \ast)
  \]

  \( \in S(1)_0 \).

- \( S(1)_d = \{ \text{trees of height } \leq d \} \), e.g. this is

  \[
  \begin{array}{c}
  \text{arrows}
  \end{array}
  \]

  \( \in S(1)_2 \).

- \( S(1)_d = \{ \text{diagrams } \ast_0 \rightarrow \ast_1 \rightarrow \cdots \rightarrow \ast_n \rightarrow \ast_0 \}

  \text{in category of finite ordinals} \)
Informal "definition" of globular operad

An ordinary operad consists of...

- for each \( n \geq 0 \), a set \( P_n \) of "\( n \)-ary operations":
  \[
  \begin{array}{c}
  \vdots \\
  \downarrow \theta \in P_n
  \end{array}
  \]

- composition:
  e.g. operations
  \[
  \begin{array}{ccc}
  \theta_1 & \circ & \theta_2 \\
  \downarrow & & \downarrow \\
  \theta_4 & \quad & \Theta
  \end{array}
  \]

  compose to give an operation
  \[
  \begin{array}{ccc}
  \vdots & \circ & \vdots \\
  \downarrow & & \downarrow \\
  \vdots & \quad & \vdots
  \end{array}
  \]

- an identity, satisfying associativity & unit laws.

A globular operad consists of...

- for each \( d \geq 0 \) and \( \pi \in \Sigma(d) \), a set \( P_{\pi} \) of "\( \pi \)-ary operations":
  \[
  \begin{array}{c}
  \quad \quad \quad \pi \\
  \downarrow \theta \in P_{\pi}
  \end{array}
  \]

- composition:
  e.g. operations
  \[
  \begin{array}{ccc}
  \theta \circ \psi \\
  \downarrow \\
  \psi
  \end{array}
  \]

  compose to give an operation
  \[
  \begin{array}{ccc}
  \quad \quad \quad \\
  \downarrow \\
  \quad \quad \quad
  \end{array}
  \]

- identities, satisfying associativity & unit laws.

Formal definition of globular operad

Key point: \( \text{GlobSet}/\mathcal{S}(1) \) is a monoidal category in an interesting way.

Given \( Q, P \in \text{GlobSet}/\mathcal{S}(1) \), define \( Q \otimes P \) by pullback:

\[
\begin{array}{c}
\text{pullback:} \\
\includegraphics{pullback_diagram.png}
\end{array}
\]

(The \( \otimes \) on \( \text{Set}/\mathcal{S}(n) \) can be described in the same way.)

Defn: A globular operad is a monoid in \( (\text{GlobSet}/\mathcal{S}(1), \otimes) \).

Any globular operad has an underlying collection, i.e., an object of \( \text{GlobSet}/\mathcal{S}(1) \).

It's a family \((P_\pi)_{d \geq 0}, \pi \in \Sigma(d)\) of sets, with sources & target maps, e.g.

\[
P_{\otimes \otimes} : P_{\ldots} 
\]
"Globular tuples"

Given a set $X$ and $n \in \mathbb{N}$, get set $X^n$ of $n$-tuples in $X$.

Given a globular set $X$ and $\pi \in S(n)$ (simplified), get set $X^\pi$ of "$\pi$-tuples in $X".

E.g.: if $\pi = (1, 2) \to (2, 1)$ in $S(2)$ then

$$X^\pi = \{\text{diagrams } x \xrightarrow{f} y \xrightarrow{h} z \text{ in } X\}$$

$$= \{(x, y, z, f, g, h, c) \in X^3 \times X^3 \times X_2 \mid s(f) = x, t(d) = y\}$$

(These are defined formally as a monoidal structure on $\text{GlobSet}/S(1)$.)

We have

$$S(X)_d = \coprod_{\pi \in S(n)} X^\pi.$$
Examples of algebras for globular operads

- The terminal globular operad 1 has $1_\pi = 1$ for all $\pi$. It’s easy to see that $T_1 \cong S$, so a 1-algebra is just a strict $\infty$-category: one operation of each globular arity.

- One strategy for giving a definition of weak $\infty$-category: choose a suitable globular operad $P$ and declare a weak $\infty$-category to be a $P$-algebra.

Then want lots of operations of each arity $\pi$.
E.g. want at least 2 elements of $P_{\pi_{1,\ldots,\pi}}$ because in a weak $\infty$-category we want $(\text{hom})$ of $\neq \text{hom}(gfh)$.

Globular operads vs. monads

For any globular operad $P$, have projection map $T_\pi \xrightarrow{\pi} S$
given by

$$(T_\pi(x))_\pi = \bigsqcup_{\pi \in \Sigma_{1,\ldots,\pi}} P_\pi \times X^\pi \xrightarrow{\pi \times \text{id}} \bigsqcup_{\pi \in \Sigma_{1,\ldots,\pi}} X^\pi = (S(x))_\pi.$$  

Pursuing this thought, discover:

- a globular operad can be defined as a cartesian monad on GlobSet equipped with a cartesian monad map to $S$ (just as for ordinary operads).

Question: Can non-isomorphic globular operads have the same underlying monad?
(Guess: no.)