The functoriality of the reflexive completion*

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Background For a small category $\mathcal{A}$, the *Isbell conjugate* of a covariant functor $Y\colon \mathcal{A} \to \mathbf{Set}$ is the contravariant functor $Y^\vee\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$Y^\vee(a) = [\mathcal{A}, \mathbf{Set}](Y, \mathcal{A}(a, -)).$$

Similarly, any functor $X\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ has a conjugate $X^\vee\colon \mathcal{A} \to \mathbf{Set}$. Conjugacy defines an adjunction

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \rightleftarrows [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}.$$

The invariant part (fixed point category) of this adjunction is called the *reflexive completion* $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$. Thus, the reflexive completion consists of those functors on $\mathcal{A}$ that are canonically isomorphic to their double conjugates. Although this definition of $\mathcal{R}(\mathcal{A})$ is only valid when $\mathcal{A}$ is small, it can be extended, with care, to arbitrary locally small categories.

The reflexive completion was introduced by Isbell as long ago as 1960 [3]. He proved two main results on it. First—modulo some important set-theoretic considerations—he characterized $\mathcal{R}(\mathcal{A})$ as the largest category into which $\mathcal{A}$ embeds as a dense and codense full subcategory. Second, he showed that reflexive completion, like Cauchy completion, is idempotent: $\mathcal{R}(\mathcal{R}(\mathcal{A})) \simeq \mathcal{R}(\mathcal{A})$ for all categories $\mathcal{A}$.

The reflexive completion of a category is related to other kinds of completion. It contains the Cauchy completion. And in a sense made precise in Proposition 12.3 of [1], it can also be described as the intersection

$$\mathcal{R}(\mathcal{A}) = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \cap [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$$

(at least when $\mathcal{A}$ is small).

Functoriality Reflexive completion is functorial (contravariantly), but only with respect to a very limited class of functors: the small-adequate ones. Here *adequate* means full, faithful, dense and codense, and the prefix *small-* refers to a condition derived from the notion of small presheaf. This functoriality of $\mathcal{R}$ follows easily from Isbell's characterization theorem (again, modulo some size considerations).

What is surprising is how often the functor $\mathcal{R}(F)\colon \mathcal{R}(\mathcal{B}) \to \mathcal{R}(\mathcal{A})$ induced by a small-adequate functor $F\colon \mathcal{A} \to \mathcal{B}$ turns out to be an equivalence. Indeed, the simplest known example where $\mathcal{R}(F)$ is *not* an equivalence is quite complex.

This phenomenon is explained in part by the following theorem. Call a category $\mathcal{A}$ *gentle* if its free completion under small colimits has small limits, and, dually, its free completion under small limits has small colimits. For example, $\mathcal{A}$ is gentle if it is small, since $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ has small limits and $[\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$ has small colimits. More substantially, Day and Lack showed that every complete and cocomplete category is gentle (Corollary 3.9 of [2]). Thus, many common categories are gentle, so that the following theorem often applies:

Theorem (Corollary 9.8 of [1]) *For every small-adequate functor $F\colon \mathcal{A} \to \mathcal{B}$ whose codomain $\mathcal{B}$ is gentle, the induced functor $\mathcal{R}(F)\colon \mathcal{R}(\mathcal{B}) \to \mathcal{R}(\mathcal{A})$ is an equivalence.*

For example, every small-adequate subcategory $\mathcal{A}$ of a complete and cocomplete category $\mathcal{B}$ has the same reflexive completion as $\mathcal{B}$. In fact, any category that is either complete *or* cocomplete is already reflexively complete (Proposition 11.11 of [1]), so then $\mathcal{R}(\mathcal{A}) \simeq \mathcal{R}(\mathcal{B}) \simeq \mathcal{B}$.

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*Joint work with Tom Avery. Abstract submitted to CT20→21.

On the other hand, it is not *always* true that $\mathcal{R}(F)$ is an equivalence, for a small-adequate functor F. I will state a theorem characterizing precisely those functors F such that $\mathcal{R}(F)$ is an equivalence. Isbell's theorem that $\mathcal{R}(\mathcal{R}(\mathcal{A})) \simeq \mathcal{R}(\mathcal{A})$ follows as a natural corollary.

If time permits, I will also mention some further results from [1]: a complete description of which (co)limits exist in every reflexive completion (destroying any hope that reflexive completion might be completion with respect to some class of limits and colimits), and results relating the reflexive completion to the Cauchy completion.

References

[1] T. Avery and T. Leinster. Isbell conjugacy and the reflexive completion. *Theory and Applications of Categories*, to appear.

[2] B. Day and S. Lack. Limits of small categories. *Journal of Pure and Applied Algebra* 210 (2007), 651–663.

[3] J. R. Isbell. Adequate subcategories. *Illinois Journal of Mathematics* 4 (1960), 541–552.