Magnitude and diversity:

How an invariant from category theory solves a problem in mathematical ecology

Tom Leinster

Glasgow/EPSRC

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2. Diversity (especially biological diversity)

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How an invariant from category theory solves a problem in mathematical ecology

1. Magnitude (the invariant from category theory)
2. Diversity (especially biological diversity)
3. How to maximize diversity (the problem and its solution)

1. Magnitude

1. Enriched categories and size-like invariants

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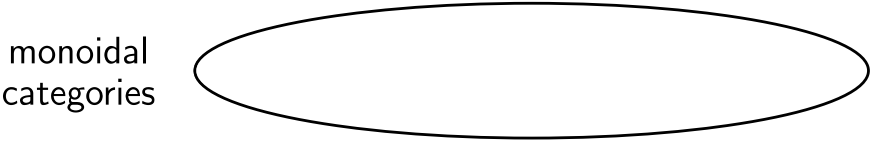
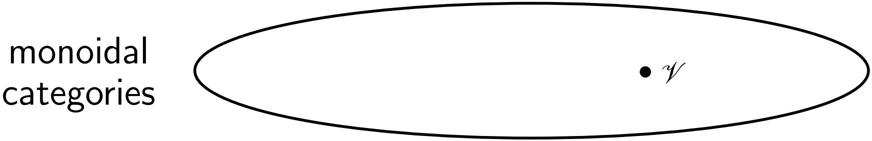


monoidal
categories

1. Enriched categories and size-like invariants



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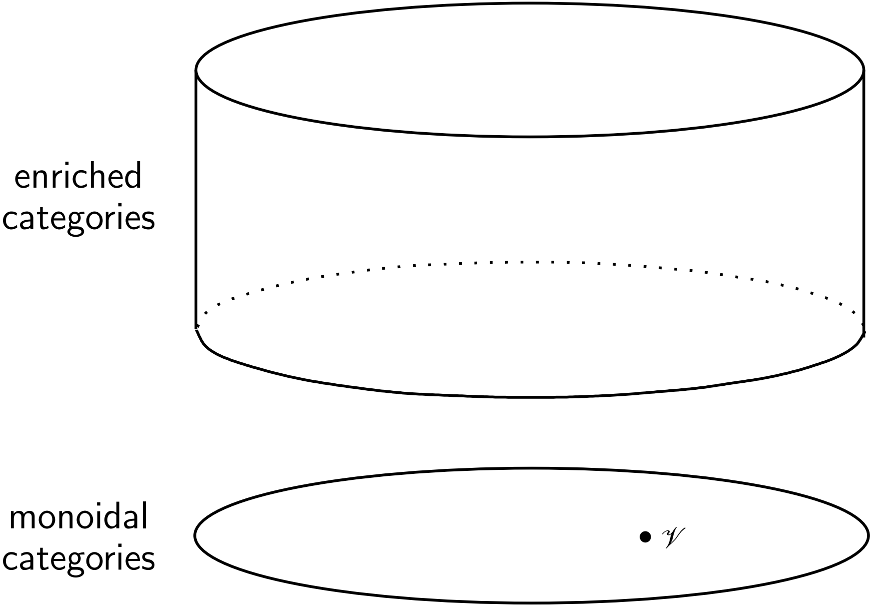

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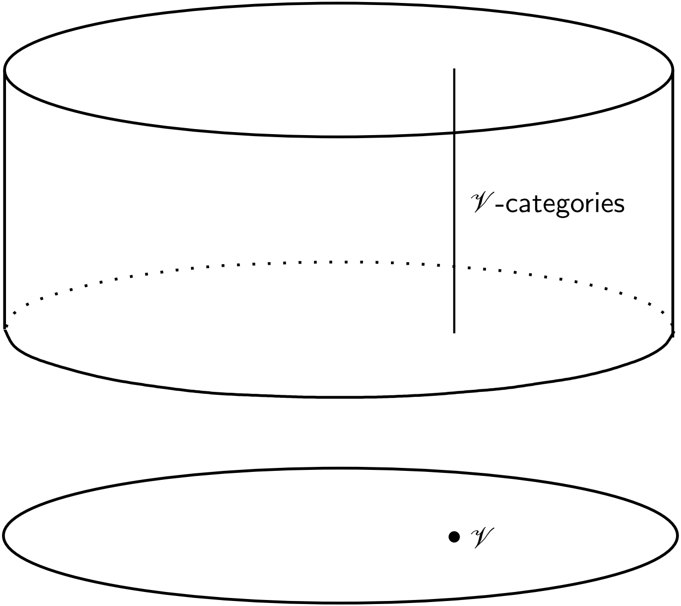

enriched
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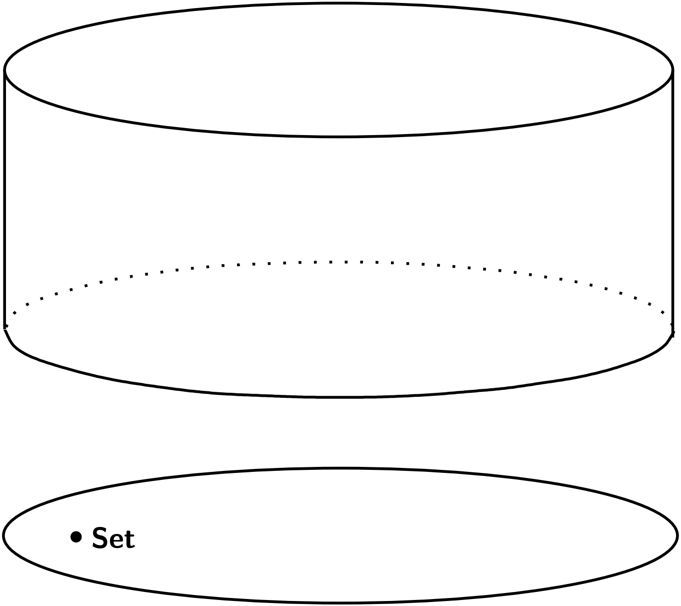

enriched
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• **Set**

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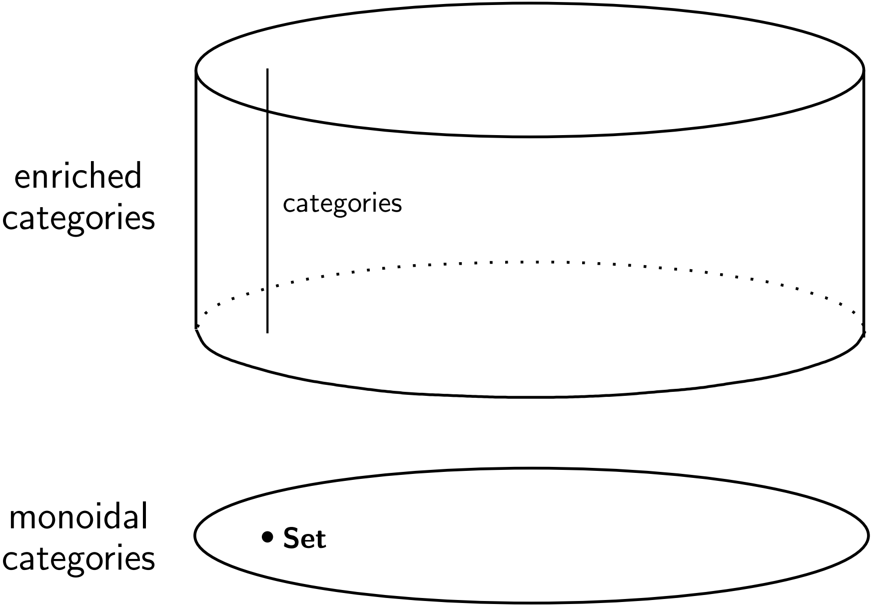

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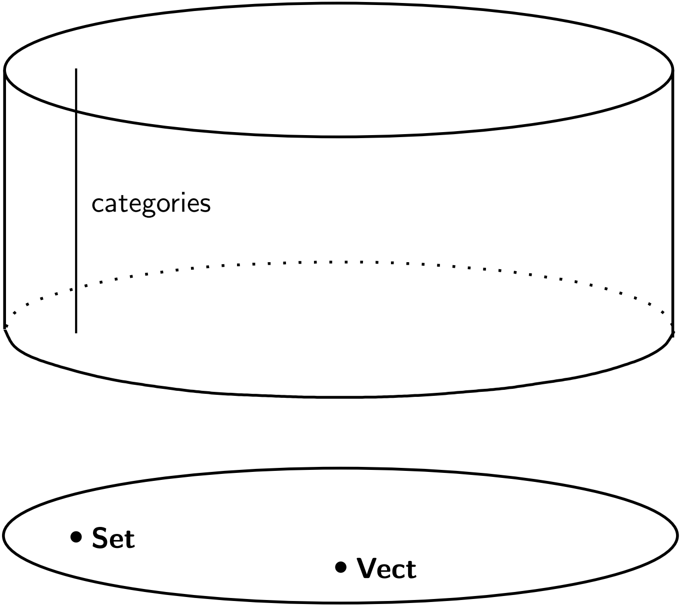

enriched
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• **Set**

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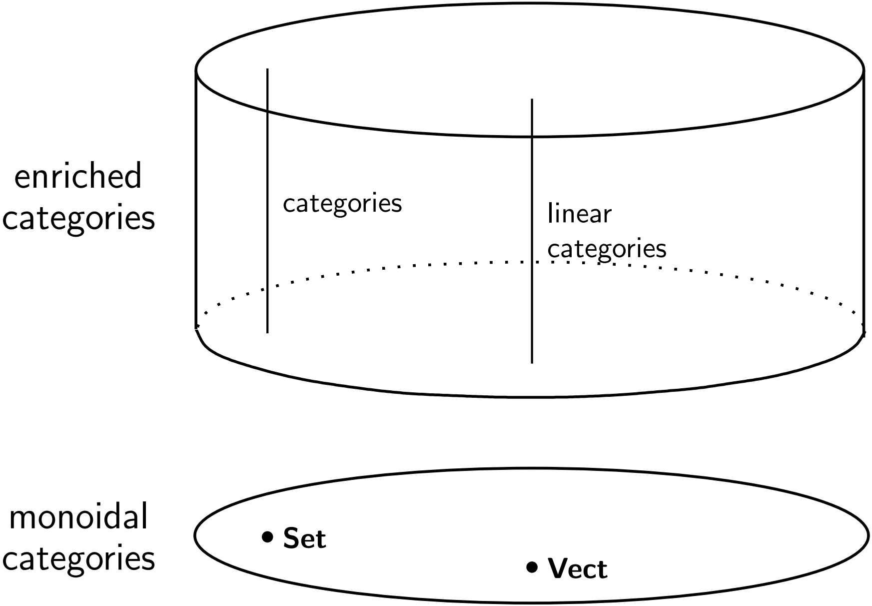

enriched
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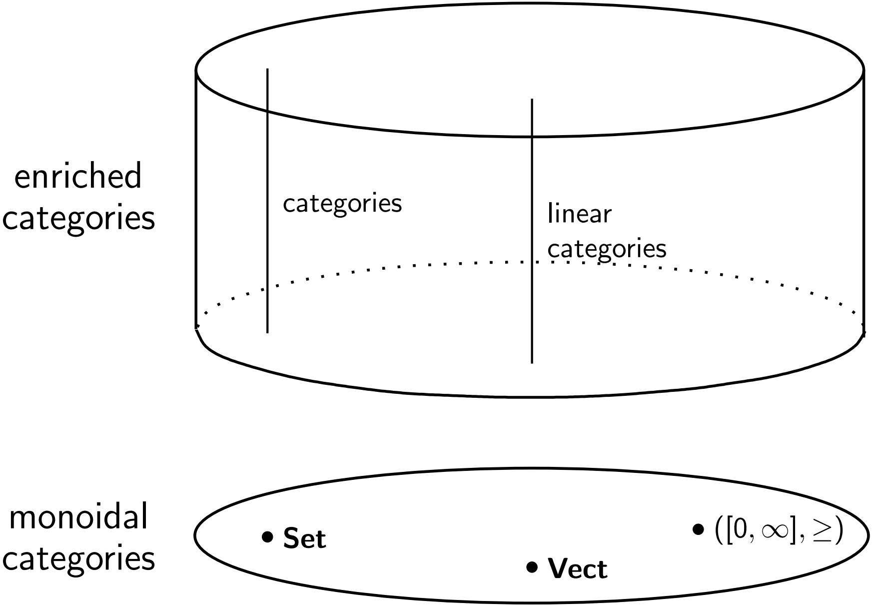

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- $([0, \infty], \geq)$

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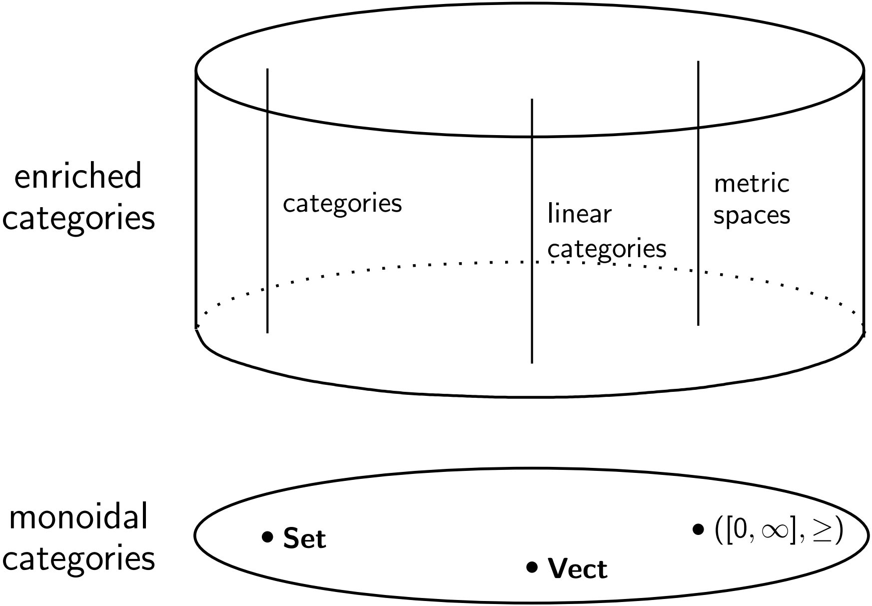

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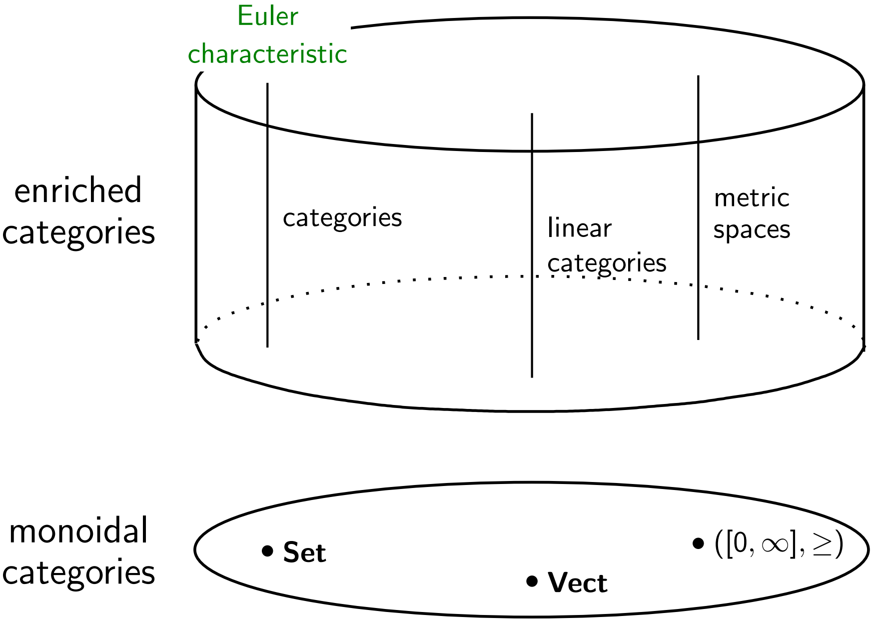

Euler characteristic

enriched categories

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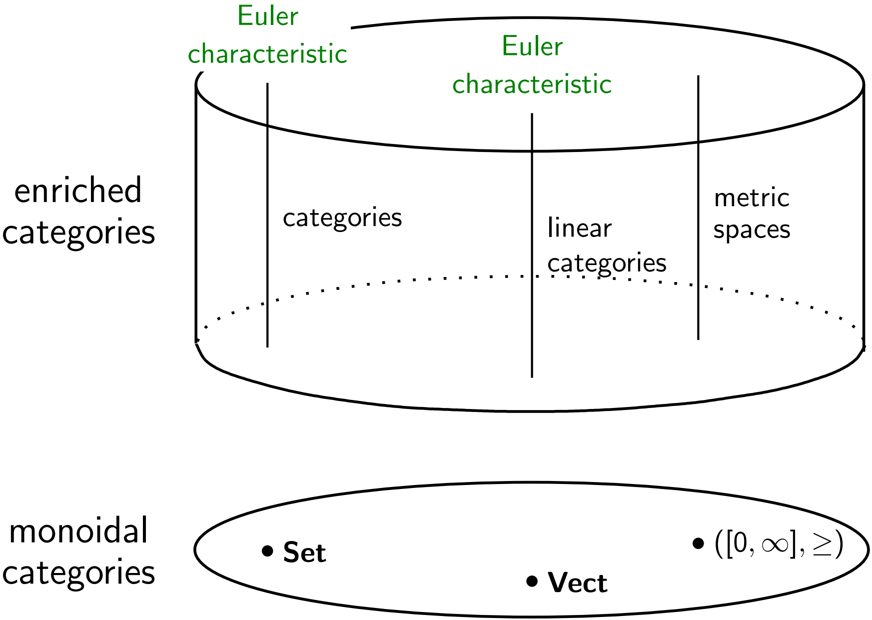

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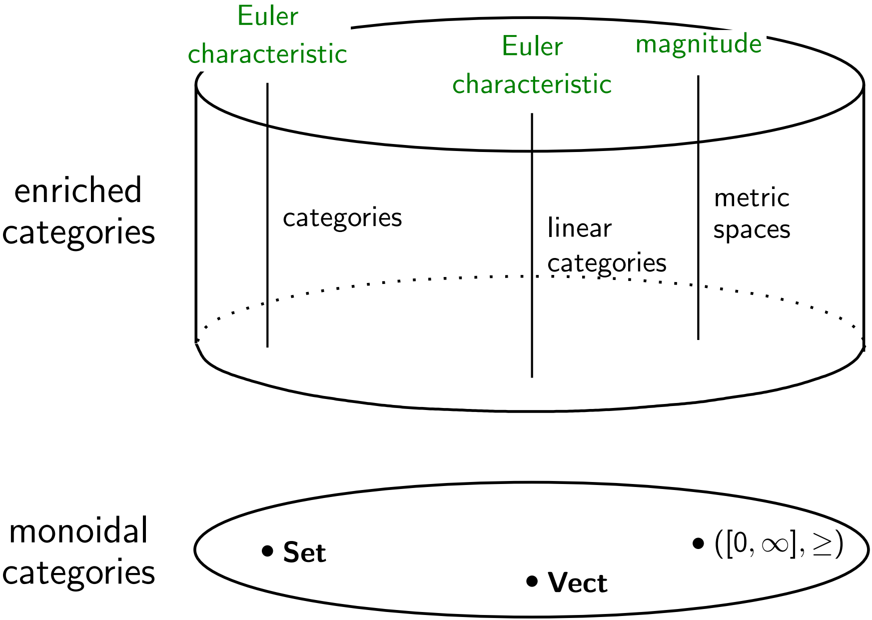

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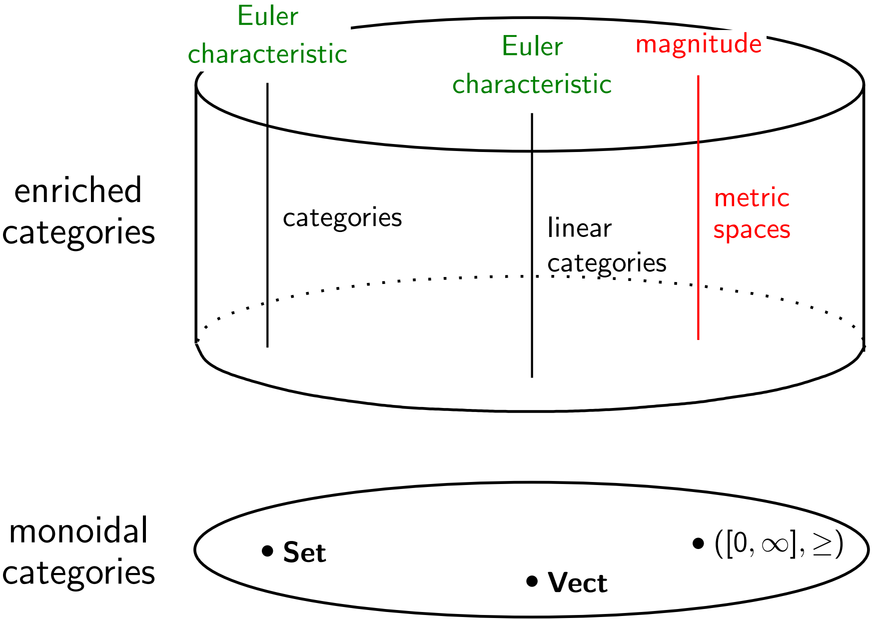

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The magnitude of a matrix

Let $\mathbf{Z}$ be a matrix.

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It is also equal to $\sum_j v_j$ for any coweighting $\mathbf{v}$.

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Then $|A| = n$.

2. Diversity

joint with Christina Cobbold (Glasgow)

A spectrum of viewpoints on biodiversity

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Conserving *species*
is what matters

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Rare species
count for as much
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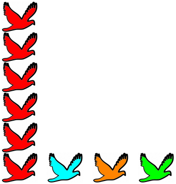

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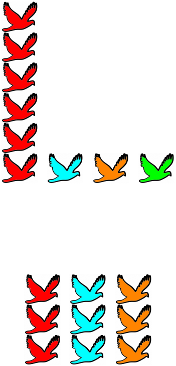

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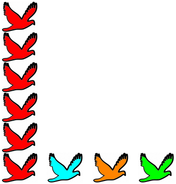

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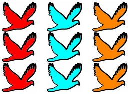

A spectrum of viewpoints on biodiversity

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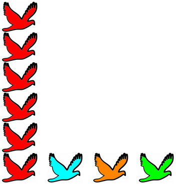

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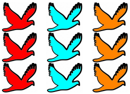

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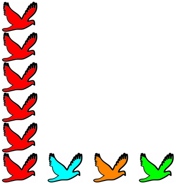

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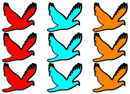

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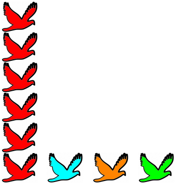

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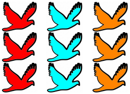

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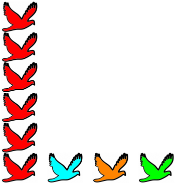

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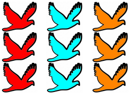


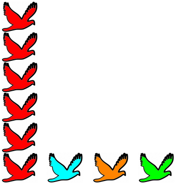
⟵ that

A spectrum of viewpoints on biodiversity

Rare species are important

Rare species are unimportant

This ⟶



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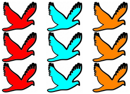

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Quantifying diversity

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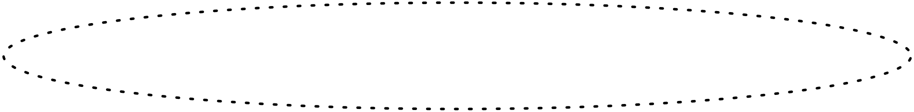

model of
community

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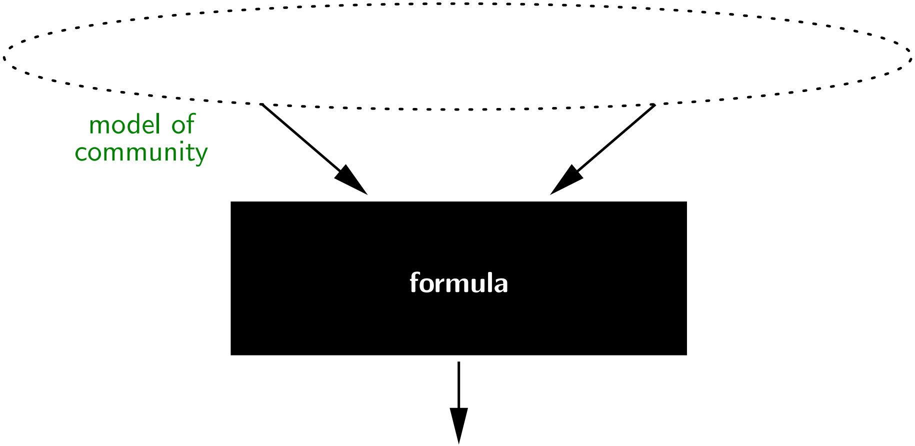

model of
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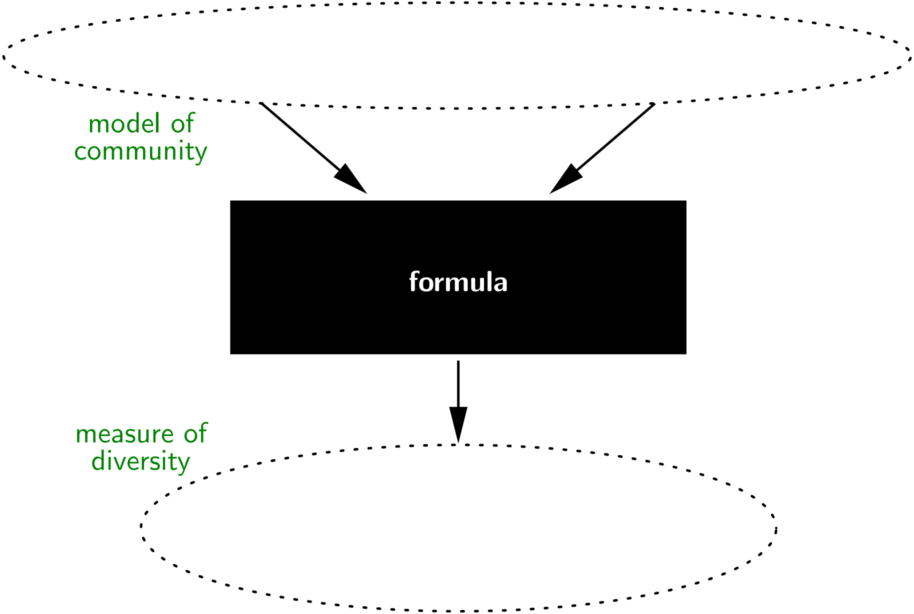

model of
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measure of
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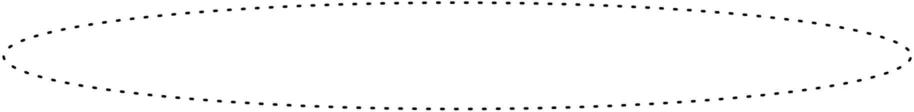

model of
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Quantifying diversity

similarity matrix **Z**

model of
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Quantifying diversity

similarity matrix $\mathbf{Z}$

Quantifying diversity

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$n \times n$ matrix ($n =$ number of species)

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$0 \leq Z_{ij} \leq 1$

Quantifying diversity

similarity matrix **Z**

$n \times n$ matrix ($n =$ number of species)

$Z_{ij} =$ similarity between ith and jth species $= Z_{ji}$

$$\underset{\substack{\text{totally}\\\text{dissimilar}}}{\underset{\uparrow}{0}} \leq Z_{ij} \leq \underset{\text{identical}}{\underset{\uparrow}{1}}$$

Quantifying diversity

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Example: a crude model would take $Z = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$.

This model assumes that distinct species are totally dissimilar.

Quantifying diversity

similarity matrix $\mathbf{Z}$

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model of
community

Quantifying diversity

similarity matrix $\mathbf{Z}$

frequency distribution $\mathbf{p}$

model of
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Quantifying diversity

similarity matrix **Z**

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Quantifying diversity

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$$\mathbf{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

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$$\mathbf{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

p_i = relative frequency,
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$p_i \geq 0$ and $\sum p_i = 1$

Quantifying diversity

similarity matrix **Z**

frequency distribution **p**

model of
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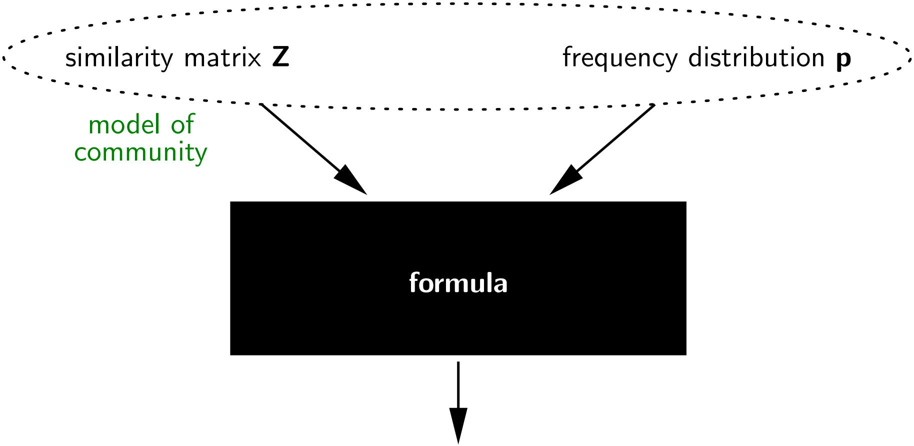
Quantifying diversity



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model of community

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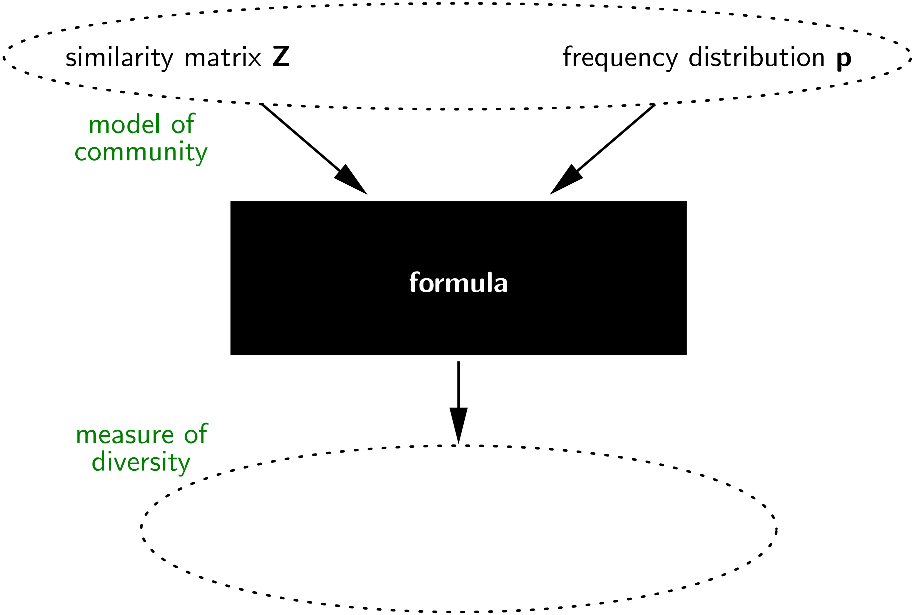

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model of
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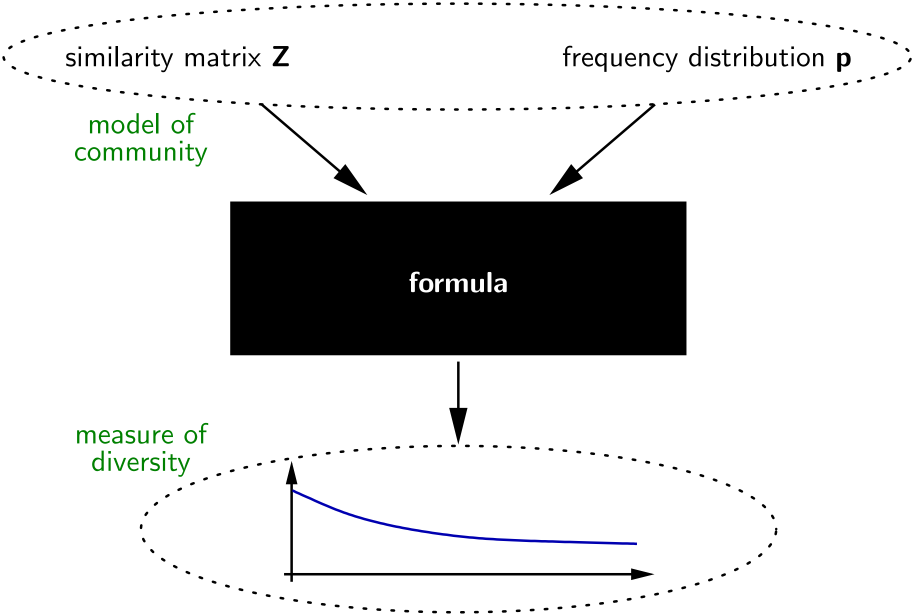
formula

measure of
diversity

Quantifying diversity



similarity matrix **Z**

frequency distribution **p**

model of community

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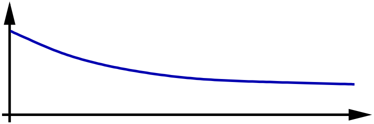

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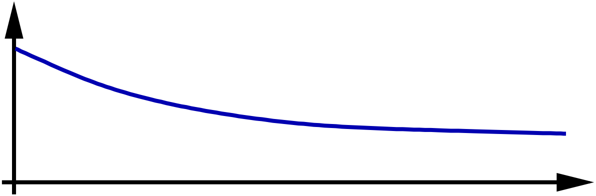

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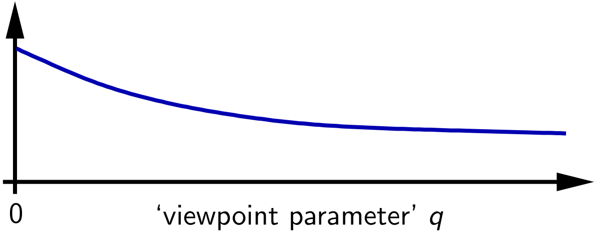

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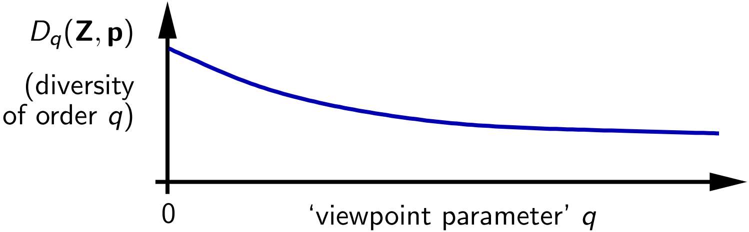

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$q = 0$:
rare species are
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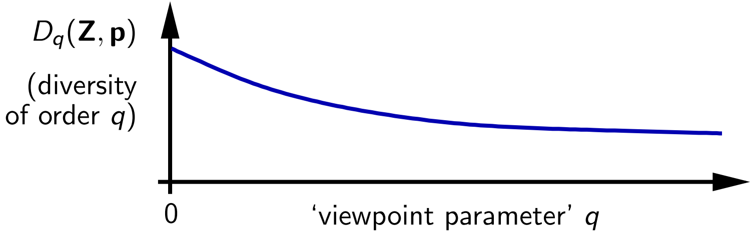

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$q = 0$:
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$q = \infty$:
rare species are
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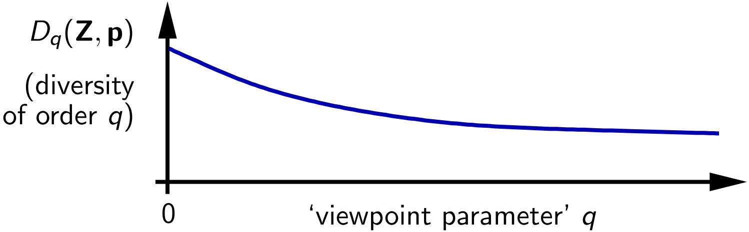

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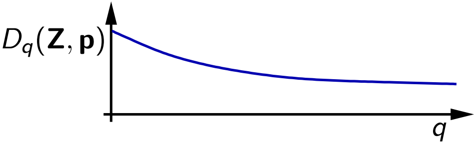

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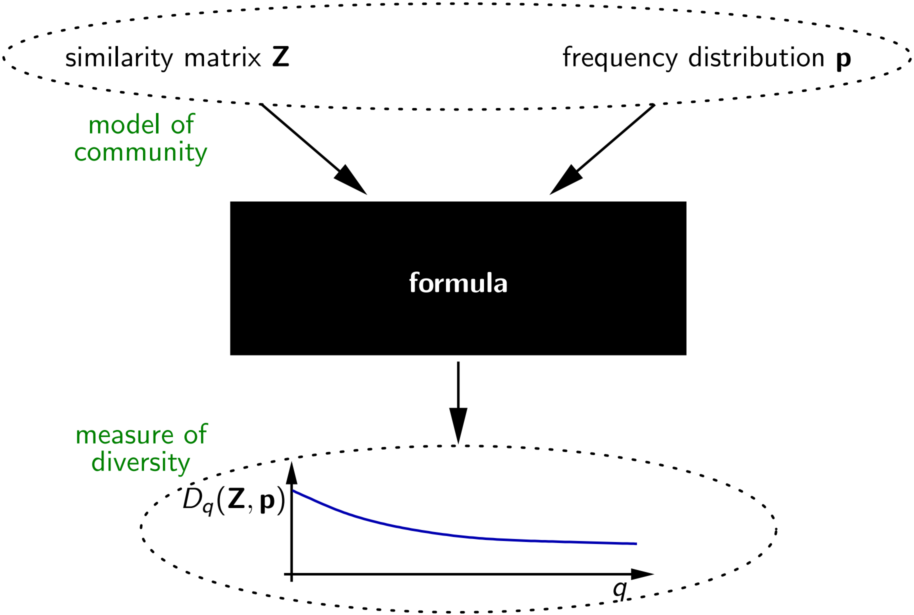

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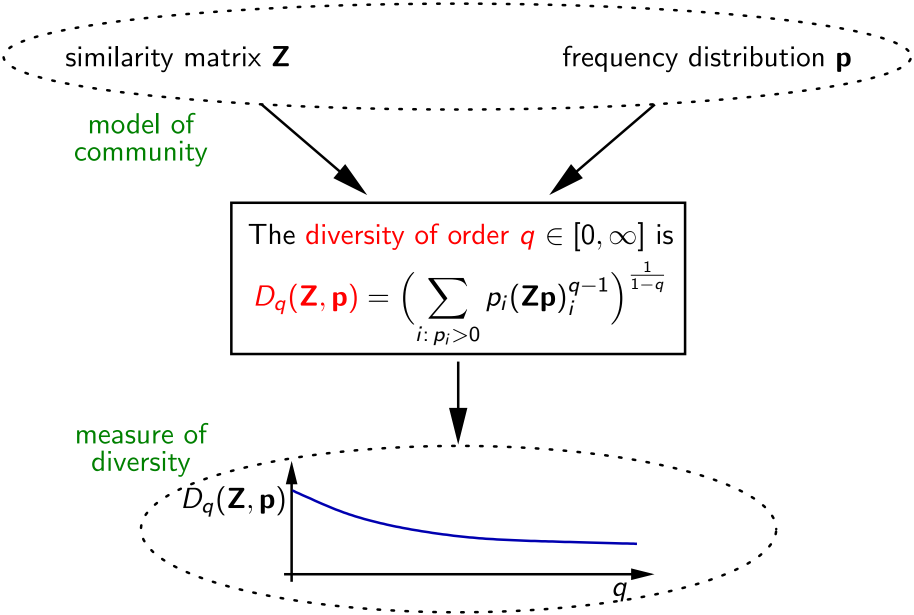

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model of community

The diversity of order $q \in [0, \infty]$ is

$$D_q(\mathbf{Z}, \mathbf{p}) = \Bigl(\sum_{i:\, p_i > 0} p_i (\mathbf{Z}\mathbf{p})_i^{q-1} \Bigr)^{\frac{1}{1-q}}$$

measure of diversity

3. How to maximize diversity

The maximum diversity problem

Fix a list of n species, with similarity matrix $\mathbf{Z}$.

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Let $0 \leq q \leq \infty$.

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I.e., what is

$$\sup\{D_q(\mathbf{Z}, \mathbf{p}) \ : \ \text{frequency distributions } \mathbf{p}\},$$

and which $\mathbf{p}$ attain this supremum?

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Also, surely the answer should mention q. . . ?
After all, different values of q represent different viewpoints on what diversity is.

The solution

Let $\mathbf{Z}$ be an $n \times n$ similarity matrix.

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Let $0 \leq q \leq \infty$*.*

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Corollary

There is a frequency distribution that maximizes diversity of order q for all q simultaneously.

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   But there is a distribution that maximizes diversity from all viewpoints simultaneously!

   ‘There is a best of all possible worlds.’