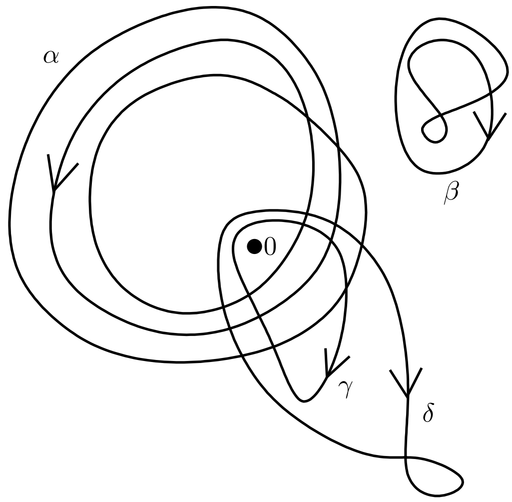
1D.11: The ‘Fundamental Theorem of Algebra’

1D.11 Theorem (‘Fundamental Theorem of Algebra’) *$\mathbb{C}$ is algebraically closed. That is, every non-constant polynomial over $\mathbb{C}$ has at least one root in $\mathbb{C}$.*

Of the many proofs, the following very intuitive one is my favourite. I will only sketch it; it can be made precise using some basic notions of algebraic topology. Evidently, it is not for examination.

First we need some general concepts. Take a path γ in $\mathbb{C}$ starting and finishing at the same point and not passing through 0. Then we may assign to



γ an integer $\#\gamma$, the **winding number** of γ, which is the number of times that γ winds anticlockwise around 0. In the figure,

$$\#\alpha = 3, \qquad \#\beta = 0, \qquad \#\gamma = -1, \qquad \#\delta = -1.$$

Moreover, suppose that γ and δ are two such paths and that (as in the figure) one can be continuously deformed to the other without passing through 0. (The technical word is **homotopy**. Imagine the plane with a pole planted at 0, and γ and δ as rubber bands.) Then $\#\gamma = \#\delta$.

Now, take a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$ of degree n and suppose that f has no complex roots. Let $r \in [0, \infty)$. As z travels one revolution anticlockwise around the circle $\{z : |z| = r\}$, $f(z)$ traces a path γ_r in $\mathbb{C} \setminus \{0\}$.

1. As r increases, γ_r changes continuously (because f is continuous), so $\#\gamma_r$ is independent of $r \in [0, \infty)$.

2. When $|z|$ is large, $f(z)$ behaves like $a_n z^n$. As the point z travels once around the circle $\{z : |z| = r\}$, the point $a_n z^n$ winds n times around 0. So for sufficiently large R, $\#\gamma_R = n$.

   (Comments: the first sentence really means that for sufficiently large R, γ_R can be continuously deformed to the path $a_n z^n$ described in the second sentence. If you have trouble seeing that this path winds n times around 0, you can assume without harm that $a_n = 1$. When $n = 3$, γ_R might look like the α of the figure.)

3. On the other hand, γ_0 stays constant at the point $f(0)$, so $\#\gamma_0 = 0$.

By (1), (2) and (3), $n = 0$. So f is constant, as required.