Rethinking set theory

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These slides: available on my web page

Anthropology of mathematicians, I

Social observations:

- Mathematicians use sets all the time
- Mathematicians rarely make mistakes in what they do with sets
- Few mathematicians could quote ‘the’ (or any) axioms for set theory.

This suggests:

> There is a reliable body of (perhaps subconscious) principles that mathematicians use when manipulating sets.

Challenge:

> Write them down.

Aim of this talk

Answer the challenge!

That is, write down a system of axioms for sets that mirrors how mathematicians actually use sets.

The answer I'll describe is due to F. William Lawvere:

- An elementary theory of the category of sets.
  Proceedings of the National Academy of Sciences of the U.S.A. **52** (1964), 1506–1511.
- An elementary theory of the category of sets (long version) with commentary.
  Reprints in Theory and Applications of Categories **12** (2005), 1–35.

In jargon, the axioms state:

> *Sets and functions form a well-pointed topos with natural numbers and choice.*

I'll state them without jargon.

Anthropology of mathematicians, II

'Sets' according to ZFC	'Sets' in ordinary usage
The elements of a set are always sets too. So, given a set X and $x \in X$, can ask 'what are the elements of x?'	The elements of a set (e.g. $\mathbb{R}$) need not be sets. 'What are the elements of π?' has no meaningful answer.
For any nonempty X, there exists $x \in X$ such that $x \cap X = \emptyset$.	This is meaningless: e.g. what is $\pi \cap \mathbb{R}$?
For any X, can form $\bigcup X$.	This is meaningless: e.g. what is $\bigcup \mathbb{R}$?

Perhaps it is misleading to use the same word, 'set', for both purposes.

Three misconceptions about categorical set theory

1. This is ‘category theory versus set theory’.
   This is not a proposed replacement for set theory. It *is* set theory.

2. Lawvere’s axioms are inherently more sophisticated than (say) ZFC.
   The axioms are completely mundane statements about sets.

3. This approach is circular.
   Actually, it’s just a first-order theory (like ZFC).

Plan for rest of talk

1. The primitive concepts
2. The axioms
3. Using the axioms as axioms

1. The primitive concepts

Elements or functions?

The working mathematician's vocabulary includes terms such as:

set, subset, element, function, equivalence relation, . . .

We need to choose some concepts as primitive and derive the others.

Usual choice of primitive concepts:

sets and elements.

Our choice:

sets and functions and composition of functions.

Elements as functions

How can we derive the concept of element from the concept of function?

Suppose for the moment that we know what a one-element set is (without knowing what an element is).

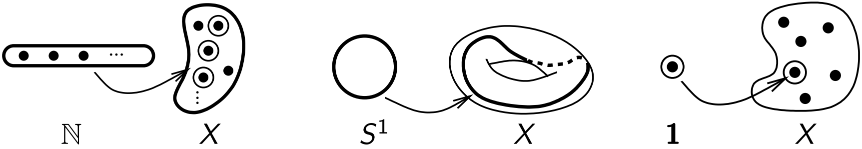
Suppose also that we have a one-element set, $\mathbf{1}$.

Then the elements of any set X are in canonical, one-to-one correspondence with the functions $\mathbf{1} \longrightarrow X$.

Why this is not a trick

There are many similar correspondences throughout mathematics, e.g.:

- a sequence in a set X is a function $\mathbb{N} \longrightarrow X$
- a loop in a topological space X is a continuous map $S^1 \longrightarrow X$
- a solution (x, y) of $x^2 + y^2 = 1$ in a ring A is a homomorphism $\mathbb{Z}[X, Y]/(X^2 + Y^2 - 1) \longrightarrow A$.



We will *define* an element of a set X to be a function $\mathbf{1} \longrightarrow X$.

2. The axioms

What we're axiomatizing

The data to which our axioms will apply:

- some things called sets
- for each set X and set Y, some things called functions from X to Y, with functions f from X to Y written as $f: X \longrightarrow Y$
- for each set X, set Y and set Z, an operation assigning to each $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ a function $g \circ f: X \longrightarrow Z$.

Informal summary of the axioms

1. Composition of functions is associative and has identities
2. There is a set with exactly one element
3. There is a set with no elements
4. A function is determined by its effect on elements
5. Given sets X and Y, one can form their cartesian product $X \times Y$
6. Given sets X and Y, one can form the set of functions from X to Y
7. Given $f : X \longrightarrow Y$ and $y \in Y$, one can form the inverse image $f^{-1}(y)$
8. The subsets of a set X correspond to the functions from X to $\{0, 1\}$
9. The natural numbers form a set
10. Every surjection has a right inverse.

Axiom 2: One-element set

Informally: 'there is a set with exactly one element'.

Definition: A set T is terminal if for all sets X, there is a unique function $X \longrightarrow T$.

Idea: The terminal sets should be exactly the one-element sets.

Axiom 2: There exists a terminal set.

Easy lemma: If T and T' are terminal, there is a unique isomorphism (i.e. invertible function) from T to T'.

So it is harmless to fix a one-element set, $\mathbf{1}$.

Definition: Let X be a set. Then $x \in X$ means $x\colon \mathbf{1} \longrightarrow X$.

Axiom 4: Functions applied to elements

Informally: 'a function is determined by its effect on elements'.

Definition: Let X be a set, $x \in X$ and $f : X \longrightarrow Y$.
We write $f(x)$ for the element $f \circ x : \mathbf{1} \longrightarrow Y$ of Y.

Axiom 4: Let X and Y be sets and $f, g : X \longrightarrow Y$ functions.
If $f(x) = g(x)$ for all $x \in X$, then $f = g$.

Axiom 9: Natural numbers

Informally: ‘the natural numbers form a set’.

Idea: We want to be able to define sequences by recursion. That is:

> whenever X is a set, $a \in X$ and $r\colon X \longrightarrow X$,
> there should be a unique sequence $(x_n)_{n=0}^{\infty}$ such that
> $x_0 = a$ and $x_{n+1} = r(x_n)$ for all $n \in \mathbb{N}$.

Definition: A natural number system is a set N together with an element $0 \in N$ and a function $s\colon N \longrightarrow N$, such that:

> whenever X is a set, $a \in X$ and $r\colon X \longrightarrow X$,
> there is a unique function $x\colon N \longrightarrow X$ such that
> $x(0) = a$ and $x(s(n)) = r(x(n))$ for all $n \in N$.

Axiom 9: There exists a natural number system.

Easy lemma: natural number systems are unique up to unique isomorphism. (So it is harmless to fix one, $\mathbb{N}$.)

3. Using the axioms as axioms

Building on the axioms

Once we've got the axioms, what next?

Sketch of development:

- Define other basic set-theoretic notions (e.g. subset).
- Show how to perform other basic set-theoretic constructions (e.g. disjoint union).

From here on, the development is the same as for many other set theories.

E.g. the construction of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ from $\mathbb{N}$ is the same as in ZFC.

Strength of the axioms

Claim (Mac Lane): The axioms suffice for almost all of mathematics.

Claim (McLarty, 2011): No more assumptions about sets are needed anywhere in EGA or SGA.

What *can't* the ten axioms do?

Simplest known example: they can't prove the existence of

$$\mathbb{N} \amalg \mathcal{P}(\mathbb{N}) \amalg \mathcal{P}(\mathcal{P}(\mathbb{N})) \amalg \cdots$$

(unless they are inconsistent).

Comparison with other set theories

Abbreviations:

- ETCS: **E**lementary **T**heory of the **C**ategory of **S**ets (the ten axioms)
- ETCS+R: add an axiom scheme of **R**eplacement
  (roughly: every family of sets $(X_i)_{i \in I}$ defined by a first-order formula has a disjoint union $\coprod_{i \in I} X_i$)
- BZC: **Z**ermelo with **B**ounded comprehension and **C**hoice.

Theorems (Cole, Lawvere, Mitchell, Osius, ...):

$$\begin{array}{ccc} \text{ETCS+R} & \longleftrightarrow & \text{ZFC} \\ \cup\!\mid & & \cup\!\mid \\ \text{ETCS} & \longleftrightarrow & \text{BZC} \end{array}$$

where $\longleftrightarrow$ means 'has same strength as'.

Should we teach ETCS instead of ZFC?

Advantages:

- Wider relevance: each axiom of ETCS has a full and active life outside set theory.
- Directly addresses the difficulties that students often have with functions.

Disadvantages:

- Has seldom been done before, so few teaching materials exist.
- Students aspiring to a career in set theory will need to learn ZFC anyway.

Thanks