Six theorems about injective metric spaces

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Introduction

A metric space Y is *injective* if every mapping which increases no distance from a subspace of any metric space X to Y can be extended, increasing no distance, over X. ARONSZAJN and PANITCHPAKDI showed [1] that topologically, every injective metric space is a complete absolute retract, and asked whether the converse is true. It is obviously true in 1-dimensional spaces. But in 2-dimensional spaces there are additional necessary conditions. First, every injective metric space can be contracted to a point *freely*, i. e. by a path $\{h_t\}$ of decreasing deformation retractions. Conversely, for 2-dimensional finite polyhedra, this condition is sufficient. It is equivalent (for any triangulation) to *collapsibility* in the sense of WHITEHEAD [5]. In infinite 2-dimensional polyhedra, collapsibility is sufficient and free contractibility necessary, and it may be that these properties are (still) equivalent.

Second topological necessary condition: a locally compact injective metric space is locally triangulable at every homotopically stable point (in the sense of HOPF and PANNWITZ [4]).

Three geometric theorems. (1) Every metric space X has a smallest containing injective *envelope* εX, which is compact if X is compact. (2) A compact injective space Y has a *boundary*, the smallest closed subset B such that $\varepsilon B = Y$. (3) An n-dimensional compact injective space has at least $2n$ boundary points and has injective n-dimensional subspaces with exactly $2n$ boundary points. Those subspaces may be chosen to be isometric copies of closed cells in n-dimensional l_∞ space.

I am indebted to T. GANEA and to W. B. WOOLF for some conversations concerning this material.

1. Polyhedra

By a *mapping* between metric spaces we mean a function $f: X \to Y$ such that for all x, x' in X, the distance $d(f(x), f(x')) \leq d(x, x')$. Y is an *injective* metric space if every mapping from a subspace of any space X to Y can be extended (to a mapping) over X. ARONSZAJN and PANITCHPAKDI introduced these spaces [1], calling them *hyperconvex* because of the characterizations which follow.