Lecture 3
Adjoints

'Adjoint functors arise everywhere' (Mac Lane).

Example 3.1 Given any set S, there is a vector space $F(S)$ with basis S; its elements can be thought of as formal linear combinations $\sum_i \lambda_i s_i$ of elements of S. It has a universal property (Example 0.2): any function from S to a vector space V extends uniquely to a linear map $F(S) \longrightarrow V$.

This can be rephrased as follows. Let $U : \mathbf{Vect} \longrightarrow \mathbf{Set}$ be the forgetful functor. Then for any set S and vector space V, there is a function

$$\text{restriction} : \mathbf{Vect}(F(S), V) \longrightarrow \mathbf{Set}(S, U(V)),$$

and the unique extension property says exactly that this function is a bijection. So we have functors $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Vect}$ such that for each $S \in \mathbf{Set}$ and $V \in \mathbf{Vect}$,

$$\mathbf{Vect}(F(S), V) \cong \mathbf{Set}(S, U(V)).$$

This isomorphism is (in a sense that can be made precise) natural in S and V.

Definition 3.2 Take categories and functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$. If

$$\mathcal{B}(FA, B) \cong \mathcal{A}(A, GB) \tag{3}$$

naturally in $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we say that F is **left adjoint** to G, and G is **right adjoint** to F, and write $F \dashv G$. An **adjunction** between F and G is a choice of natural isomorphism (3).

What this says is that for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, a map $FA \longrightarrow B$ is the same thing as a map $A \longrightarrow GB$. The correspondence is written as $\overline{(\ )}$ (**transpose**) in both directions:

$$\begin{array}{ccc} (FA \xrightarrow{g} B) & \longmapsto & (A \xrightarrow{\overline{g}} GB), \\ (FA \xrightarrow{\overline{f}} B) & \longleftarrow\!\!\shortmid & (A \xrightarrow{f} GB), \end{array}$$

and $\overline{\overline{f}} = f$, $\overline{\overline{g}} = g$. In the examples that follow, I'll be content to interpret the 'naturally' part of the definition informally, but for the record, naturality amounts to the equation

$$\overline{\left(FA' \xrightarrow{Fp} FA \xrightarrow{g} B \xrightarrow{q} B'\right)} = \left(A' \xrightarrow{p} A \xrightarrow{\bar{g}} GB \xrightarrow{Gq} GB'\right).$$

A given functor G may not have a left adjoint, but if it does, it is unique up to isomorphism, so we may speak of '*the* left adjoint of G' (and dually for right adjoints). We prove this uniqueness next lecture.

Example 3.3 (Free $\dashv$ forgetful) Forgetful functors between categories of algebras always have left adjoints:

$$\begin{tikzcd}
\mathbf{Vect} \arrow[d, shift left, "U_1"] \\
\mathbf{Set} \arrow[u, shift left, "F_1"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Gp} \arrow[d, shift left, "U_2"] \\
\mathbf{Set} \arrow[u, shift left, "F_2"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Ring} \arrow[d, shift left, "U_3"] \\
\mathbf{Monoid} \arrow[u, shift left, "F_3"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{AssAlg} \arrow[d, shift left, "U_4"] \\
\mathbf{LieAlg} \arrow[u, shift left, "F_4"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Ab} \arrow[d, shift left, "U_5"] \\
\mathbf{Gp} \arrow[u, shift left, "F_5"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Gp} \arrow[d, shift left, "U_6"] \\
\mathbf{Monoid} \arrow[u, shift left, "F_6"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}.$$

- $F_1 \dashv U_1$ is the adjunction of Example 3.1.
- $F_2 \dashv U_2$ is very similar; F_2 forms the free group on a set.
- U_3 forgets the additive structure of a ring, and

  $$F_3(M) = \mathbb{Z}M = \{\text{formal sums } \textstyle\sum \lambda_i m_i \text{ with } \lambda_i \in \mathbb{Z},\ m_i \in M\}$$

(which when M is a group is called the **group ring** of M).

- U_4 forgets everything about an associative algebra B except its underlying vector space and its commutator; thus $U_4(B)$ is a Lie algebra with bracket $[b_1, b_2] = b_1 b_2 - b_2 b_1$. The left adjoint F_4 assigns to a Lie algebra its **universal enveloping algebra**.

- U_5 forgets that an abelian group is abelian; its left adjoint F_5 is abelianization (Example 1.13), and adjointness says that a map $G \longrightarrow A$ is the same thing as a map $G^{\mathrm{ab}} \longrightarrow A$ (for $G \in \mathbf{Gp}$ and $A \in \mathbf{Ab}$).

- U_6 forgets that a group has inverses; its left adjoint F_6 turns a monoid M into a group by throwing in a formal inverse m^{-1} for each element m of M. This left adjoint is crucial in K-theory.

Warning: categories such as (fields) and (torsion abelian groups) are not categories of algebras in the sense used here, and their forgetful functors to **Set** do not have left adjoints.

Example 3.4 ($\otimes$ and Hom) For modules A, B, and C over a commutative ring k, there is an isomorphism

$$k\text{-}\mathbf{Mod}(A \otimes B, C) \cong k\text{-}\mathbf{Mod}(A, \mathbf{Hom}_k(B, C))$$

where $\mathbf{Hom}_k(B, C)$ is the set $k\text{-}\mathbf{Mod}(B, C)$ with its evident k-module structure. So for every k-module B gives rise to an adjunction

$$\begin{tikzcd}
k\text{-}\mathbf{Mod} \arrow[d, shift left=2, "\mathbf{Hom}_k(B,-)"] \arrow[d, phantom, "\dashv"] \\
k\text{-}\mathbf{Mod} \arrow[u, shift left=2, "- \otimes B"]
\end{tikzcd}.$$

A slightly different situation is 'change of base': any map $\phi : k \longrightarrow k'$ of commutative rings induces functors and adjunctions

$$\begin{tikzcd}
k\text{-}\mathbf{Mod} \arrow[d, shift right=5, "k' \otimes_k -"'] \arrow[d, shift left=5, "\mathbf{Hom}_k(k',-)"] \arrow[d, phantom, shift right=2.5, "\dashv"] \arrow[d, phantom, shift left=2.5, "\dashv"] \\
k'\text{-}\mathbf{Mod}. \arrow[u, "\phi^*"]
\end{tikzcd}$$

Example 3.5 (Topology) Miscellaneous topological adjunctions:

$$\begin{array}{ccc} \mathbf{Top} & \mathbf{CptHff} & \mathbf{CompleteMet} \\ D \left\uparrow \dashv \; U \downarrow \; \dashv \right\uparrow I & F_7 \left\uparrow \dashv \right\downarrow U_7 & F_8 \left\uparrow \dashv \right\downarrow U_8 \\ \mathbf{Set} & \mathbf{Top} & \mathbf{Met}. \end{array}$$

- U sends a topological space to its underlying set, and D and I endow a set with the discrete and indiscrete topology, respectively.
- U_7 is the inclusion of compact Hausdorff spaces into all spaces, and F_7 is Stone-Čech compactification (more on which below).
- **Met** is the category of metric spaces and uniformly continuous maps, **CompleteMet** is the full subcategory consisting of the complete metric spaces, U_8 is inclusion, and F_8 is metric space completion.

Example 3.6 (Galois connections) Take ordered sets A and B and an adjoint pair

$$\begin{array}{c} B^{\mathrm{op}} \\ f \Big\uparrow \dashv \Big\downarrow g \\ A. \end{array}$$

This means that f and g are order-reversing functions and

$$b \leq f(a) \iff a \leq g(b).$$

Such an adjunction $f \dashv g$ is called a **Galois connection** because of the following example: for fields $K \subseteq M$, we have

$$\begin{array}{c} (\text{subgroups of } \mathrm{Gal}(M : K))^{\mathrm{op}} \\ \text{invariant subgroup} \Big\uparrow \dashv \Big\downarrow \text{fixed field} \\ (\text{intermediate fields of } M : K). \end{array}$$

For any Galois connection $f \dashv g$ there is an order-reversing isomorphism

$$(\mathrm{image}(f))^{\mathrm{op}} \cong \mathrm{image}(g).$$

In the example given (Galois theory), the existence of the Galois connection is trivial; identifying the images is the hard part, and to do it we usually make simplifying assumptions such as normality and separability.

$*$ $*$ $*$

There are two other ways of saying what an adjunction is. First:

Proposition 3.7 *Take categories and functors* $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$*. Then there is a one-to-one correspondence between*

- *adjunctions* $F \dashv G$
- *pairs* $(1_{\mathcal{A}} \xrightarrow{\eta} G \circ F,\ F \circ G \xrightarrow{\varepsilon} 1_{\mathcal{B}})$ *of natural transformations satisfying the* **triangle identities***: the triangles*

$$\begin{tikzcd} F(A) \arrow[r, "F\eta_A"] \arrow[dr, "1_{F(A)}"'] & FGF(A) \arrow[d, "\varepsilon_{F(A)}"] \\ & F(A) \end{tikzcd} \qquad \begin{tikzcd} G(B) \arrow[r, "\eta_{G(B)}"] \arrow[dr, "1_{G(B)}"'] & GFG(B) \arrow[d, "G\varepsilon_B"] \\ & G(B) \end{tikzcd}$$

commute for all $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$*.*

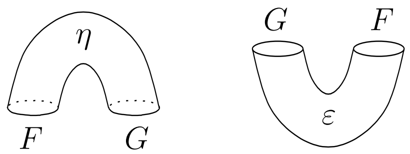
Sketch proof An adjunction $F \dashv G$ consists of a bijection

$$\mathcal{B}(FA, B) \xrightarrow{\sim} \mathcal{A}(A, GB)$$

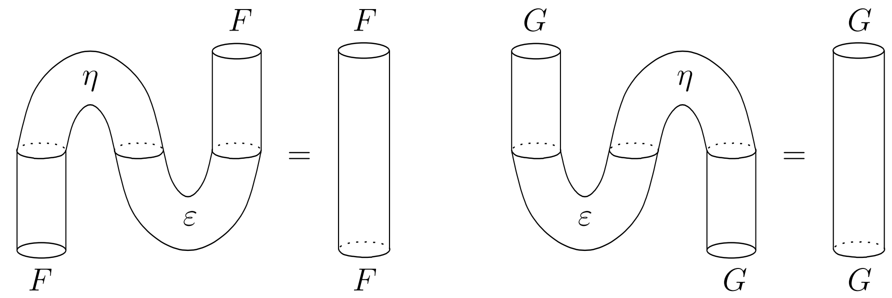
for each A and B, satisfying naturality axioms. For each $A \in \mathcal{A}$ we can take $B = FA$ and the identity $1_{FA} \in \mathcal{B}(FA, FA)$ to obtain a map $\eta_A = \overline{1_{FA}}$: $A \longrightarrow GFA$. This defines a natural transformation $\eta : 1 \longrightarrow GF$, the **unit** of the adjunction. Dually, there is a **counit** $\varepsilon : FG \longrightarrow 1$. The triangle identities follow from naturality.

Conversely, the whole adjunction can be recovered from the unit and counit: with the usual notation, $\overline{g} = (Gg) \circ \eta_A$ and $\overline{f} = \varepsilon_B \circ (Ff)$. $\square$

Digression 3.8 The unit and counit can sensibly be drawn as



and the triangle identities then become the topologically plausible



Moreover, suppose you're in some category such as $\mathbf{Vect}_k$ that comes with a tensor product $\otimes$ and a unit object k for the tensor: then you can sensibly define a **(left) dual** of an object V as an object V^* together with maps $\eta : k \longrightarrow V \otimes V^*$ and $\varepsilon : V^* \otimes V \longrightarrow k$ satisfying equations like the triangle identities. A nice explanation of these connections is in Joachim Kock, *Frobenius Algebras and 2D Topological Quantum Field Theories*, Cambridge, 2003.

So far, both formulations of adjointness look symmetric. But the third, which you'll probably see most often, does not.

Example 3.9 Consider once again the universal property of a vector space $F(S)$ with basis S. There is a map $\eta_S : S \longrightarrow U(F(S))$ picking out the basis elements, and the 'unique extension property' mentioned previously can be drawn as

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & U(F(S)) \arrow[d, "U(\exists! \overline{f})"] \\ & U(V). \end{tikzcd}$$

That is, for any vector space V and function f from S to (the underlying set of) V, there is a unique linear map $\overline{f} : F(S) \longrightarrow V$ extending f to $F(S)$.

This statement turns out to be precisely equivalent to adjointness. In general:

Proposition 3.10 *The following are equivalent for a functor $G : \mathcal{B} \longrightarrow \mathcal{A}$:*

- *G has a left adjoint*
- *for each $A \in \mathcal{A}$, there exist an object $FA \in \mathcal{B}$ and a map $\eta_A : A \longrightarrow G(FA)$ with the universal*

property

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & G(FA) \arrow[d, "G(\exists!\overline{f})"] \\ & G(B). \end{tikzcd}$$

Sketch proof If G has a left adjoint F then the universal property follows (eventually) from naturality of the adjunction. Conversely, suppose we have an object FA and a map η_A for each $A \in \mathcal{A}$, with the given universal property. Then F can be made into a functor in a unique way such that η_A is a natural transformation, and the universal property tells us that for any A and B, the function

$$\begin{array}{ccc} \mathcal{B}(FA, B) & \longrightarrow & \mathcal{A}(A, GB) \\ g & \longmapsto & (Gg) \circ \eta_A \end{array}$$

is bijective. So F is left adjoint to G. □

Often right adjoints are easily described but the left adjoint is a lot of trouble. There are existence theorems of the form 'any functor satisfying certain conditions has a left adjoint'—and of course, that left adjoint is unique when it exists.

Let $G : \mathcal{B} \longrightarrow \mathcal{A}$ be a functor between locally small categories, and suppose that $\mathcal{B}$ has all limits and G preserves them. (This terminology will be defined later.)

Theorem 3.11 (General Adjoint Functor Theorem) *If G satisfies certain further conditions then G has a left adjoint.*

This implies that forgetful functors between categories of algebras, such as those in Example 3.3, always have left adjoints.

Theorem 3.12 (Special Adjoint Functor Theorem) *If $\mathcal{B}$ satisfies certain further conditions then G has a left adjoint.*

This gives, for instance, Stone-Čech compactification (Example 3.5). The Theorem does not merely imply that the left adjoint exists, but actually constructs it: it tells us that the Stone-Čech compactification of a space A is the closure of the image of the canonical map $A \longrightarrow [0,1]^{\mathbf{Top}(A,[0,1])}$.

Exercises

3.13 Find three more examples of adjunctions.

3.14 Let G be a group.

a. What interesting functors are there (in either direction) between **Set** and the category $[G, \mathbf{Set}]$ of sets with a left G-action? What adjunctions are there between these functors?

(Hints: before you dive into it formally, ask yourself 'given a set with a G-action, what sets arise from it?' and *vice versa*.

I have six functors and four adjunctions in mind. It might help to observe that $\mathbf{Set} \cong [1, \mathbf{Set}]$, where 1 is the trivial group, and to know that for any functor $T : \mathbb{A} \longrightarrow \mathbb{B}$ between small categories, the functor

$$\begin{array}{rccc} T^* : & [\mathbb{B}, \mathbf{Set}] & \longrightarrow & [\mathbb{A}, \mathbf{Set}] \\ & Y & \longmapsto & Y \circ T \end{array}$$

always has both a left and a right adjoint.)

b. What happens if we replace **Set** by $\mathbf{Vect}_k$, so that we're looking for adjunctions between $\mathbf{Vect}_k$ and the category $[G, \mathbf{Vect}_k]$ of k-linear representations of G?