Solutions to exercises from Lecture 2

2.19

a. Given a map $f : X \longrightarrow Y$ in $\mathcal{B}$, define $\mathbf{Sym}(f) : \mathbf{Sym}(X) \longrightarrow \mathbf{Sym}(Y)$ by $p \longmapsto f \circ p \circ f^{-1}$. This makes $\mathbf{Sym}$ into a functor since $(g \circ f) \circ p \circ (g \circ f)^{-1} = g \circ (f \circ p \circ f^{-1}) \circ g^{-1}$ and $1 \circ p \circ 1^{-1} = p$.

   Given a map $f : X \longrightarrow Y$ in $\mathcal{B}$, define $\mathbf{Ord}(f) : \mathbf{Ord}(X) \longrightarrow \mathbf{Ord}(Y)$ by $R \longmapsto f_* R$ where, if R is a total order on X, $f_* R$ is the total order on Y given by

$$(y, y') \in f_* R \iff (f^{-1}y, f^{-1}y') \in R.$$

   This is clearly functorial.

b. Suppose that there is a natural transformation $\alpha : \mathbf{Sym} \longrightarrow \mathbf{Ord}$. Then for any bijection $f : X \longrightarrow Y$ of finite sets, the square

$$\begin{tikzcd}
\mathbf{Sym}(X) \arrow[r, "\mathbf{Sym}(f)"] \arrow[d, "\alpha_X"'] & \mathbf{Sym}(Y) \arrow[d, "\alpha_Y"] \\
\mathbf{Ord}(X) \arrow[r, "f_*"'] & \mathbf{Ord}(Y)
\end{tikzcd}$$

   commutes. Note that $(\mathbf{Sym}(f))(1_X) = f \circ 1_X \circ f^{-1} = 1_Y$: so if we take $Y = X$ and write R for the total order $\alpha_X(1_X)$ on X then the square says that $f_* R = R$. In particular, this holds when X is a 2-element set $\{x_1, x_2\}$ and f is the non-identity permutation of X, so $(x_1, x_2) \in R \iff (x_2, x_1) \in R$. Since R is a total order, this is impossible.

Part (b) shows that $\mathbf{Sym}$ and $\mathbf{Ord}$ are not naturally isomorphic. But if X is a finite set then there are the same number of bijections $X \longrightarrow X$ and total orders on X (namely, $|X|!$), so $\mathbf{Sym}(X) \cong \mathbf{Ord}(X)$.

2.20 Conjugacy.

Let $H \underset{q}{\overset{p}{\rightrightarrows}} G$ be homomorphisms between monoids, or equivalently functors between one-object categories. Writing $\star$ for the single object of both categories, a natural transformation $\alpha : p \longrightarrow q$ consists of a map $\alpha_\star$ in the category G such that for all maps h in the category H, the square

$$\begin{tikzcd}
\star \arrow[r, "p(h)"] \arrow[d, "\alpha_\star"'] & \star \arrow[d, "\alpha_\star"] \\
\star \arrow[r, "q(h)"'] & \star
\end{tikzcd}$$

commutes. In other words, it consists of an element g of the monoid G such that $g \cdot p(h) = q(h) \cdot g$ for all $h \in H$. When G and H are groups, this says that $q(h) = gp(h)g^{-1}$ for all $h \in H$. So two group homomorphisms are naturally isomorphic if and only if they are, in this sense, conjugate.

Now take $H = \mathbb{Z}$ and let $p, q : \mathbb{Z} \longrightarrow G$ be the homomorphisms corresponding to elements $x, y \in G$ respectively; thus $p(n) = x^n$ and $q(n) = y^n$. By the above, p and q are naturally isomorphic if and only if there exists $g \in G$ such that $y^n = gx^ng^{-1}$ for all $n \in \mathbb{Z}$, or equivalently $y = gxg^{-1}$.

2.21 Define $F : \mathbf{Mat} \longrightarrow \mathbf{FDVect}$ on objects by $n \longmapsto k^n$, and on maps by the usual matrix-multiplication formula: the image under F of an $n \times m$ matrix M is the linear map

$$\begin{array}{ccc}
k^m & \longrightarrow & k^n \\
(x_1, \ldots, x_m) & \longmapsto & (\sum_i M_{1i}x_i, \ldots, \sum_i M_{ni}x_i).
\end{array}$$

You're meant to guess that the composition and identities in the category **Mat** are given by matrix multiplication and the identity matrix, and it is straightforward to show that F is indeed a functor.

The standard correspondence between $n \times m$ matrices and linear transformations $k^m \longrightarrow k^n$ says precisely that F is full and faithful. The fact that every finite-dimensional vector space is isomorphic to k^n for some $n \in \mathbb{N}$ says that F is essentially surjective on objects. So $F : \mathbf{Mat} \longrightarrow \mathbf{FDVect}$ is an equivalence.

This functor F is canonical, in the sense that it can be defined without making any random choices. But consider what a pseudo-inverse to F (that is, a functor $G : \mathbf{FDVect} \longrightarrow \mathbf{Mat}$ satisfying $G \circ F \cong 1$ and $F \circ G \cong 1$) would have to be: we would have to put $G(V) = \dim V$, and then to define G on maps, we would have to choose a basis for every finite-dimensional vector space, and this cannot be done canonically.