Solutions to exercises from Lecture 3

3.13 A few possibilities:

- The forgetful functor

$$\textbf{Field} \longrightarrow (\text{integral domains } + \text{injective homomorphisms})$$

  has a left adjoint, sending an integral domain to its field of fractions (Example 0.1).

- The forgetful functor $\textbf{Gp} \longrightarrow \textbf{Monoid}$ also has a *right* adjoint, sending a monoid to its group of invertible elements.

- The forgetful functor from (topological monoids) to **Top** has a left adjoint F, which forms the free topological monoid on a space. $F(A)$ is the disjoint union $\coprod_{n\geq 0} A^n$ of the finite powers of A, that is, the space of finite sequences in A; multiplication is concatenation of sequences.

- For any field k and $n \in \mathbb{N}$ there is a Galois connection

$$\begin{tikzcd} (\text{ideals of } k[X_1, \ldots, X_n])^{\mathrm{op}} \arrow[d, shift left, "\mathcal{V}"] \\ (\text{subsets of } k^n) \arrow[u, shift left, "\mathcal{I}"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

  (for which see any introduction to algebraic geometry).

- For any topological space X there is a category $\textbf{Pshf}(X) = [\textbf{Open}(X)^{\mathrm{op}}, \textbf{Set}]$ of presheaves on X, and a full subcategory $\textbf{Sh}(X)$ consisting of the sheaves. The inclusion $\textbf{Sh}(X) \longrightarrow \textbf{Pshf}(X)$ has a left adjoint, called **sheafification** or the **associated sheaf functor**.

- There is a functor $S : \textbf{Top} \longrightarrow [\Delta^{\mathrm{op}}, \textbf{Set}]$ assigning to a space X its singular simplicial set $S(X)$, given by $S(X)_n = \textbf{Top}(\Delta^n, X)$. It has a left adjoint R, **geometric realization**.

3.14

a. There are unique group homomorphisms $G \longrightarrow 1$ and $1 \longrightarrow G$, which can be regarded as functors between one-object categories. As in the hint, composition with these gives functors $\mathbf{Set} \longrightarrow [G, \mathbf{Set}]$ and $[G, \mathbf{Set}] \longrightarrow \mathbf{Set}$ respectively, and by the hint again, each of these has adjoints on each side. The adjunctions are as follows:

$$
\begin{tikzcd}
{[G,\mathbf{Set}]} \arrow[d, bend right=60, "(\ )_G"'] \arrow[d, bend left=60, "(\ )^G"] \arrow[d, phantom, "\dashv\ \ \dashv"] \\
\mathbf{Set}. \arrow[u, "\Delta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
{[G,\mathbf{Set}]} \arrow[d, "U"] \arrow[d, phantom, "\dashv\ \ \dashv"] \\
\mathbf{Set} \arrow[u, bend left=60, "G \times -"] \arrow[u, bend right=60, "\mathbf{Hom}(G,-)"']
\end{tikzcd}
$$

I will describe the first few functors and adjunctions in detail, and the rest more sketchily.

- $\Delta : [1, \mathbf{Set}] \longrightarrow [G, \mathbf{Set}]$ is composition with the unique functor $G \longrightarrow 1$. In other words, it sends a set A to the same set A with trivial G-action.
- Let B be a G-set (i.e., a set equipped with a G-action). A map $B \longrightarrow \Delta A$ of G-sets is a function $\phi : B \longrightarrow A$ such that $\phi(gb) = g\phi(b)$ for all $g \in G$, $b \in B$; but the action of G on A is trivial, so this says that $\phi(gb) = \phi(b)$ for all g, b. So such a map is just a function $B \longrightarrow A$ constant on each orbit of B; in other words, it is a function from $B_G = \{G\text{-orbits of } B\}$ to A.
- Again, let B be a G-set. A map $\Delta A \longrightarrow B$ of G-sets is a function $\phi : A \longrightarrow B$ such that $\phi(ga) = g\phi(a)$ for all g, a, that is, $\phi(a) = g\phi(a)$ for all g, a. So such a map is just a function from A to the set B^G of fixed points of the G-action on B.
- U simply forgets the G-action on a set.
- Every set A gives rise to a set $G \times A$ with G-action $g \cdot (h, x) = (gh, x)$.
- Every set A also gives rise to a set $\mathbf{Set}(G, A)$ with G-action given by $(h\alpha)(g) = \alpha(gh)$ ($g, h \in G$, $\alpha \in \mathbf{Set}(G, A)$). I write this G-set as $\mathbf{Hom}(G, A)$. (Compare Example 3.4.)

b. This looks much the same when $\mathbf{Set}$ is replaced by $\mathbf{Vect}_k$. In the first pair of adjunctions we have

$$B_G = B/(\text{subspace generated by } \{gb - b \mid g \in G, b \in B\}),$$

and in the second, $\times$ changes to $\otimes$ and $\mathbf{Hom}$ uses linear maps.