Solutions to exercises from Lecture 6

6.19 By Theorem 6.5, it is enough to show that none of these forgetful functors preserves colimits. In fact, none of them even preserves initial objects. For each of the categories **Gp**, k-**Mod**, **LieAlg**, and **Ring** does have an initial object (respectively: $\{1\}$, $\{0\}$, $\{0\}$, and $\mathbb{Z}$), and in each case the (underlying set of the) initial object is nonempty. But the initial object of **Set** is $\emptyset$.

6.20 Take a category $\mathcal{A}$, a diagram $D : \mathbb{I} \longrightarrow \mathcal{A}$ that has a colimit, and an object $A \in \mathcal{A}$. The functor D gives rise to a functor $D^{\mathrm{op}} : \mathbb{I}^{\mathrm{op}} \longrightarrow \mathcal{A}^{\mathrm{op}}$, and a colimit of D in $\mathcal{A}$ is exactly a limit of D^{op} in $\mathcal{A}^{\mathrm{op}}$, so D^{op} has a limit. Equation (7) applied to $\mathcal{A}^{\mathrm{op}}$, A and D^{op} is

$$\mathcal{A}^{\mathrm{op}}(A, \lim_{\leftarrow I} D^{\mathrm{op}} I) \cong \lim_{\leftarrow I} \mathcal{A}^{\mathrm{op}}(A, D^{\mathrm{op}} I),$$

that is,

$$\mathcal{A}(\lim_{\rightarrow I} DI, A) \cong \lim_{\leftarrow I} \mathcal{A}(DI, A),$$

as required.

6.21 Briefly put: epics are characterized by colimits, so epics in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ are computed pointwise; hence α is epic exactly when each α_A is surjective.

More slowly: let $\alpha : X \longrightarrow Y$ be a map in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ (thus, a natural transformation). Then α is epic if and only if the square

$$\begin{tikzcd} & Y \arrow[dr, "1"] & \\ X \arrow[ur, "\alpha"] \arrow[dr, "\alpha"'] & & Y \\ & Y \arrow[ur, "1"'] & \end{tikzcd}$$

is a pushout, by Proposition 5.7. Corollary 6.14 tells us that pushouts in

$[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ are computed pointwise, so this square is a pushout if and only if

$$\begin{tikzcd}
& YA \arrow[dr, "1"] & \\
XA \arrow[ur, "\alpha_A"] \arrow[dr, "\alpha_A"'] & & YA \\
& YA \arrow[ur, "1"'] &
\end{tikzcd}$$

is a pushout for each $A \in \mathbb{A}$. By Proposition 5.7 again, this is true if and only if each α_A is epic. But epics in **Set** are just surjections, so α is epic if and only if each α_A is surjective.

I do not know of any more direct proof, although there is one for the analogous fact about monics.