Solutions to exercises from Lecture 7

7.22 Loosely, a monad on $\mathcal{A}$ is a functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ together with a way of associating to each $n \in \mathbb{N}$ precisely one natural transformation $T^n \longrightarrow T$.

Loosely, a T-algebra is an object A of $\mathcal{A}$ together with a way of associating to each $n \in \mathbb{N}$ precisely one map $T^n A \longrightarrow A$.

7.23 Some possibilities:

- Most trivially, a functor is non-monadic if it has no left adjoint. A non-trivial example is the forgetful functor $U : \mathbf{TAb} \longrightarrow \mathbf{Set}$ on the category of torsion abelian groups (those in which every element has finite order). Suppose that $F \dashv U$. The unit map $\eta_1 : 1 \longrightarrow UF1$ picks out an element u of $F1$ and has the universal property

$$\begin{tikzcd} 1 \arrow[r, "\eta_1"] \arrow[dr, "a"'] & UF1 \arrow[d, dotted, "U(\exists! g)"] \\ & UA \end{tikzcd}$$

—for all $A \in \mathbf{TAb}$ and $a \in A$, there is a unique homomorphism $g : F1 \longrightarrow A$ such that $g(u) = a$. So for all torsion abelian groups A and all $a \in A$, the order of a divides the order of u. But for any $n \geq 1$ we may take $A = \mathbb{Z}/n\mathbb{Z}$ and a to be the generator of A, so n divides the order of u for all $n \geq 1$, a contradiction.

- The forgetful functor $U : \mathbf{Poset} \longrightarrow \mathbf{Set}$ on the category of ordered sets does have a left adjoint, but is not monadic. The left adjoint D orders a set A by equality: $a \leq a' \iff a = a'$. There is a map f in $\mathbf{Poset}$ from $D\{0, 1\}$ to ($\{0, 1\}$ with its usual ordering) defined by $f(i) = i$. Then Uf is an isomorphism but f is not, so U does not reflect isomorphisms, so is not monadic.

- The functor $\mathrm{ob} : \mathbf{Cat} \longrightarrow \mathbf{Set}$ assigning to a small category its set of objects is not monadic, as it is not faithful: consider one-object categories, for instance.

- The forgetful functor $U : \mathbf{TFAb} \longrightarrow \mathbf{Set}$ on the category of torsion-free abelian groups (those in which every non-zero element has infinite order) has a left adjoint, is faithful, and reflects isomorphisms, but is not monadic. The left adjoint F sends a set A to the free abelian group $\mathbb{Z}A$ on A, so the monad $\mathbb{T} = (T, \mu, \eta)$ induced by $F \dashv U$ is the same as the monad induced by the adjunction $\mathbf{Ab} \rightleftarrows \mathbf{Set}$. Hence $\mathbf{Set}^{\mathbb{T}} \simeq \mathbf{Ab}$, and it suffices to show that $\mathbf{TFAb} \not\simeq \mathbf{Ab}$.

If I had given you the precise definition of monadicity then you'd be able to see that in fact, it suffices to show that the inclusion $\mathbf{TFAb} \hookrightarrow \mathbf{Ab}$ is not an equivalence; and this is immediate as the inclusion is not essentially surjective on objects. In any case, there is no equivalence whatsoever between $\mathbf{TFAb}$ and $\mathbf{Ab}$. Consider the following property of a category: 'there exists an object with exactly two endomorphisms'. This property is invariant under equivalence. It holds for $\mathbf{Ab}$, as witnessed by the group with two elements. But it does not hold for $\mathbf{TFAb}$. For let $A \in \mathbf{TFAb}$: then $a \longmapsto na$ defines an endomorphism of A for each $n \in \mathbb{N}$, and torsion-freeness implies that these endomorphisms are all distinct unless A is trivial.

7.24 A group is a set A equipped with a binary operation m (usually written $\cdot$), a unary operation i (usually written $(\ )^{-1}$), and a nullary operation (constant) 1, satisfying axioms

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad x \cdot 1 = x = 1 \cdot x, \qquad x \cdot x^{-1} = 1 = x^{-1} \cdot x$$

$(x, y, z \in A)$. (You may be used to thinking of a group as a set equipped with a binary operation such that there *exist* a unit element and inverses, and of course the unit and inverses are then unique. But for these purposes it is better to think of the unit and inverses as structure rather than properties: thus, on the same footing as the multiplication.)

Now define a 'Freyd–Heller algebra' to be a set A equipped with m, i, 1 as above, a unary operation f, and a nullary operation α, satisfying the axioms above and the further axioms

$$f(x \cdot y) = f(x) \cdot f(y), \qquad f(f(x)) = \alpha^{-1} \cdot f(x) \cdot \alpha$$

$(x, y \in A)$. The theory of Freyd–Heller algebras is just like any other algebraic theory, so the category $\mathbf{FH}$ of Freyd–Heller algebras is monadic over $\mathbf{Set}$. By the final paragraph of the lecture, $\mathbf{FH}$ therefore has an initial object (F, f, α), and initiality is exactly the universal property described in the quote.

(Alternatively, the forgetful functor $\mathbf{FH} \longrightarrow \mathbf{Set}$ has a left adjoint, and since $\mathbf{Set}$ has an initial object and left adjoints preserve initial objects, $\mathbf{FH}$ also has an initial object.)

F is, incidentally, the much-studied **Thompson group**.