Solutions to exercises from Lecture 5

5.14 Take an ordered set $(A, \leq)$, a set I, and a family $(a_i)_{i \in I}$ of elements of A. A cone on the diagram $(a_i)_{i \in I}$ is an element $l \in A$ with inequalities $l \leq a_i$ for each $i \in I$, that is, a lower bound of $\{a_i \mid i \in I\}$. It is a product cone if whenever z is also a lower bound, $z \leq l$. So the product $\prod_{i \in I} a_i$, if it exists, is the **greatest lower bound** or **inf** or **meet** of $\{a_i \mid i \in I\}$, often written $\bigwedge_{i \in I} a_i$.

Dually, $\coprod_{i \in I} a_i$ is the **least upper bound** or **sup** or **join** $\bigvee_{i \in I} a_i$.

Examples: if A is a power-set and $\leq = \subseteq$ then $\bigwedge$ is intersection and $\bigvee$ is union. If $A = \mathbb{N}$ and $a \leq b$ means $a|b$ then products and coproducts of *finite* families exist; they are greatest common divisor and lowest common multiple, respectively.

5.15 By the formula in 5.4, $C = S/\sim$ where $\sim$ is the equivalence relation on S generated by $s \sim f(s)$ for all $s \in S$. Explicitly, $s_1 \sim s_2 \iff \exists m, n \in \mathbb{N} : f^m(s) = f^n(s)$. The topology on $S/\sim$ is the quotient topology: a subset of $S/\sim$ is open just when its inverse image under the quotient map $S \longrightarrow S/\sim$ is open.

If f is a bijection then $s_1 \sim s_2 \iff \exists n \in \mathbb{Z} : s_1 = f^n(s_2)$, so each equivalence class is countable and C is uncountable. Let f be rotation of S^1 by an irrational number of revolutions: then every equivalence class is dense, so the closure of every point of C is C itself, so C is indiscrete.

5.16 The first 'special case' in Example 5.5 says that monics are stable under pullback in **Set**, since any monic is (to within isomorphism) inclusion of a subset. However, we have to prove this in an arbitrary category.

Take a pullback square

$$\begin{tikzcd}
 & A_1 \arrow[dr, "f_1"] & \\
P \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & B \\
 & A_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

and suppose that f_2 is monic. Take $Z \underset{v}{\overset{u}{\rightrightarrows}} P$ such that $p_1 u = p_1 v$. We have

$$f_2 p_2 u = f_1 p_1 u = f_1 p_1 v = f_2 p_2 v$$

and f_2 is monic, so $p_2 u = p_2 v$. The outer part of the square

$$\begin{tikzcd}
 & & A_1 \arrow[dr, "f_1"] & \\
Z \arrow[r, dotted] \arrow[urr, "p_1 u = p_1 v"] \arrow[drr, "p_2 u = p_2 v"'] & P \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & & B \\
 & & A_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

certainly commutes, so there is a unique dotted arrow making the inner triangles commute. But the triangles commute when the dotted arrow is either u or v, so $u = v$.

Take maps $A \underset{m}{\overset{e}{\rightleftarrows}} B$ satisfying $em = 1_B$; we have to prove that e is epic. Indeed, if $B \underset{v}{\overset{u}{\rightrightarrows}} C$ with $ue = ve$ then $u = u1_B = uem = vem = v1_B = v$.

(The **Axiom of Choice** is that every epic in **Set** is split. In this sense, the Axiom of Choice fails in most categories of algebras. For instance, it fails in **Ab** because the epic $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is not split: 'we cannot choose a section in the world of abelian groups'.)