Lecture 7
Monads

'A monad is an algebraic theory.'

Definition 7.1 Let $\mathcal{A}$ be a category. A **monad** on $\mathcal{A}$ is a functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ together with natural transformations

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "T \circ T"{name=U}] \arrow[r, bend right, "T"'{name=D}] & \mathcal{A}, \arrow[from=U, to=D, Rightarrow, "\mu"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "1"{name=U}] \arrow[r, bend right, "T"'{name=D}] & \mathcal{A} \arrow[from=U, to=D, Rightarrow, "\eta"]
\end{tikzcd}
$$

(the **multiplication** and **unit** of the monad) satisfying associativity and unit axioms: for each $A \in \mathcal{A}$, the diagrams

$$
\begin{tikzcd}
 & T^3A \arrow[dl, "\mu_{TA}"'] \arrow[dr, "T\mu_A"] & \\
T^2A \arrow[dr, "\mu_A"'] & & T^2A \arrow[dl, "\mu_A"] \\
 & TA &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, "\eta_{TA}"'] \arrow[dd, "1"] \\
T^2A \arrow[dr, "\mu_A"'] & \\
 & TA
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[dr, "T\eta_A"] \arrow[dd, "1"'] & \\
 & T^2A \arrow[dl, "\mu_A"] \\
TA &
\end{tikzcd}
$$

commute.

Remarks 7.2 A monad is a type of monoid, as we'll see next lecture: hence the name. Monads were once called 'triples' (because a monad is a 3-tuple (T, μ, η)). Some people still call them that.

Example 7.3 Fix a commutative ring k. There is a monad (T, μ, η) on **Set** defined as follows.

- Let A be a set. Then TA is the set of formal k-linear combinations $\sum_{i \in I} \lambda_i a_i$ (I finite, $\lambda_i \in k$, $a_i \in A$). (To be precise, a 'formal k-linear combination' of elements of A is a family $(\lambda_a)_{a \in A} \in k^A$ such that $\lambda_a = 0$ for all but finitely many $a \in A$.)

- $\mu_A : T(TA) \longrightarrow TA$ sends a formal linear combination of formal linear combinations to a single formal linear combination:
$$\sum_{i \in I} \left(\sum_{j \in J_i} \lambda_{ij} a_{ij} \right) \longmapsto \sum_{i \in I, j \in J_i} (\lambda_i \lambda_{ij}) a_{ij}.$$

- $\eta_A : A \longrightarrow TA$ sends $a \in A$ to the trivial linear combination $\sum_{i \in 1} 1 \cdot a$.

Example 7.4 There is a monad (T, μ, η) on **Set** defined by:

- $TA = \{$group-style words in A, e.g. $a_1^2 a_2^{-1} a_1 a_2^3\}$.
- $\mu_A : T^2 A \longrightarrow TA$ 'removes brackets', e.g.

$$(a_1 a_2^2 a_3^{-1})^{-1} (a_1 a_4)^2 \longmapsto a_3 a_2^{-2} a_4 a_1 a_4$$

- $\eta_A : A \longrightarrow TA$ sends $a \in A$ to the trivial word a.

Example 7.5 Any adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[d, shift left, "U"] \\ \mathcal{A} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

gives rise to a monad on $\mathcal{A}$. Write η and ε for the unit and counit of the adjunction. Then the monad is given by $T = U \circ F : \mathcal{A} \longrightarrow \mathcal{A}$, $\eta = \eta$, and

$$\mu_A = U\varepsilon_{FA} : U(FU(FA)) \longrightarrow U(FA).$$

The monad axioms follow from the triangle identities (3.7).

For instance, the previous two examples come from taking $\mathcal{A} = \mathbf{Set}$ and $\mathcal{B} = k$-**Mod** or **Gp**.

We will see that *every* monad arises from an adjunction. But very different adjunctions can give rise to the same monad: for instance,

$$\begin{tikzcd} \mathbf{Top} \arrow[d, shift left, "U"] \\ \mathbf{Set} \arrow[u, shift left, "D"] \arrow[u, phantom, "\dashv"] \end{tikzcd}, \qquad \begin{tikzcd} \mathbf{Set} \arrow[d, shift left, "1"] \\ \mathbf{Set} \arrow[u, shift left, "1"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

(where D sends a set to the corresponding discrete space) both induce the identity monad on **Set**.

Definition 7.6 Let $T = (T, \mu, \eta)$ be a monad on a category $\mathcal{A}$. A T**-algebra** is an object $A \in \mathcal{A}$ together with a map $h : TA \longrightarrow A$ such that the diagrams

$$\begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "Th"'] & TA \arrow[d, "h"] \\ TA \arrow[r, "h"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1"'] & TA \arrow[d, "h"] \\ & A \end{tikzcd}$$

commute. A **homomorphism** $(A, h) \longrightarrow (B, k)$ of

T-algebras is a map $f : A \longrightarrow B$ in $\mathcal{A}$ such that

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "h"'] & TB \arrow[d, "k"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

commutes. The category of T-algebras and homomorphisms is written $\mathcal{A}^T$.

Example 7.7 Let T be the monad on **Set** from Example 7.3. Then a T-algebra is a set A together with a function

$$h : \{\text{formal } k\text{-linear combinations of elements of } A\} \longrightarrow A,$$

satisfying axioms. Thus, a T-algebra is just a k-module. (This should seem plausible; we'll see how best to prove it later.)

For instance, any $a_1, a_2 \in A$ give rise to a formal linear combination $\sum_{i\in\{1,2\}} a_i \in TA$, and addition in the k-module is given by $a_1 + a_2 = h(\sum_{i\in\{1,2\}} a_i)$.

Example 7.8 Similarly, if T is the group monad (7.4) then a T-algebra is a set A together with a function

$$h : \{\text{group-style words in } A\} \longrightarrow A$$

satisfying axioms, and this is just a group.

Note that in the monadic approach, the operations $(a, b) \longmapsto a \cdot b$ and $(a, b) \longmapsto a^{-1}ba^2$ have exactly the same status—they are both just elements of TA. As usual, category theory makes no arbitrary choices.

Example 7.9 Take an adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[d, shift left, "U"] \\ \mathcal{A} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \tag{10}$$

and the induced monad $T = U \circ F$ on $\mathcal{A}$ (7.5). In the previous two examples, $\mathcal{A}^T \simeq \mathcal{B}$, but this is not always the case: for instance, if $\mathcal{A} = \mathbf{Set}$ and $\mathcal{B} = \mathbf{Top}$ as in 7.5 then $\mathcal{A}^T \cong \mathbf{Set} \not\simeq \mathbf{Top} = \mathcal{B}$. Roughly, an adjunction (10) is 'monadic' if $\mathcal{A}^T \simeq \mathcal{B}$. So $\mathbf{Top} \rightleftarrows \mathbf{Set}$ is not monadic; this makes precise the feeling that topology is not a branch of algebra.

* * *

In this course we have seen that forgetful functors between categories of algebras, such as

$$
\begin{tikzcd}
k\text{-}\mathbf{Mod} \arrow[d] & \mathbf{Gp} \arrow[d] & \mathbf{Ab} \arrow[d] & \mathbf{AssAlg} \arrow[d] \\
\mathbf{Set}, & \mathbf{Set}, & \mathbf{Gp}, & \mathbf{LieAlg},
\end{tikzcd}
\tag{11}
$$

have certain common properties: for instance, they all have left adjoints and, therefore, preserve limits. In the rest of this lecture we describe precisely what this class of functors is, and see how to prove that a functor does or does not belong to it.

Construction 7.10 Let $T = (T, \mu, \eta)$ be a monad on a category $\mathcal{A}$. Define functors

$$
\begin{tikzcd}
\mathcal{A}^T \arrow[d, shift left=2, "U^T"] \\
\mathcal{A} \arrow[u, shift left=2, "F^T"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

by

$$U^T\begin{pmatrix} TA \\ \downarrow h \\ A \end{pmatrix} = A, \qquad F^T(A) = \begin{pmatrix} T^2A \\ \downarrow \mu_A \\ TA \end{pmatrix}.$$

Proposition 7.11 *$F^T \dashv U^T$, and the induced monad on $\mathcal{A}$ is T.*

Proof Omitted, except to note that $U^T \circ F^T = T$. □

Corollary 7.12 *Every monad is induced by some adjunction.* □

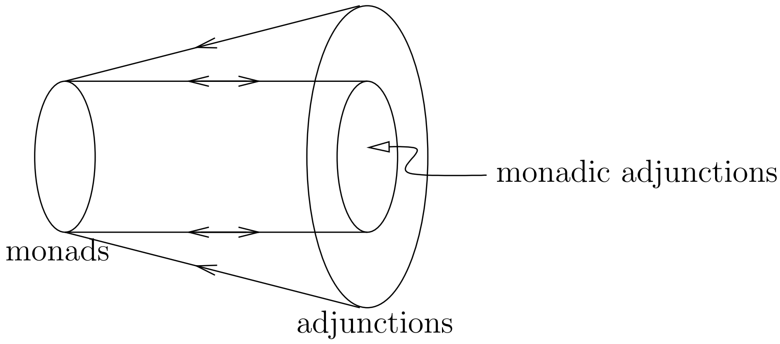
Definition 7.13 An adjunction is **monadic** if, up to equivalence, it is of the form

$$\begin{tikzcd} \mathcal{A}^T \arrow[d, shift left, "U^T"] \\ \mathcal{A} \arrow[u, shift left, "F^T"] \end{tikzcd} \quad F^T \dashv U^T.$$

(This is slightly imprecise, as I have not said what 'up to equivalence' means.) A functor U is **monadic** if it has a left adjoint F and the adjunction $F \dashv U$ is monadic.

If $\mathcal{A}$ and $\mathcal{B}$ are categories for which there is an obvious forgetful functor $U : \mathcal{B} \longrightarrow \mathcal{A}$, we sometimes say that $\mathcal{B}$ **is monadic over** $\mathcal{A}$ to mean that U is monadic. For instance, **Gp** is monadic over **Set** (as we will see).

Picture:



$$
T \qquad \longmapsto \qquad
\begin{tikzcd}
\mathcal{A}^T \arrow[d, shift left=2, "U^T"] \\
\mathcal{A} \arrow[u, shift left=2, "F^T"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

$$
U \circ F \qquad \longleftarrow\!\shortmid \qquad
\begin{tikzcd}
\mathcal{B} \arrow[d, shift left=2, "U"] \\
\mathcal{A} \arrow[u, shift left=2, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

Problem: how can you show that a given functor $U : \mathcal{B} \longrightarrow \mathcal{A}$ is monadic? To show directly that $\mathcal{A}^T \simeq \mathcal{B}$ (where T is the monad induced by U and its left adjoint F), you need to know what F is, and typically

F is hard to describe. Indeed, you may only know of the existence of F by an adjoint functor theorem.

Solution: the Monadicity Theorem gives necessary and sufficient conditions for U to be monadic, *without reference to F.*

Theorem 7.14 (Monadicity) *A functor U is monadic if and only if it has a left adjoint and creates coequalizers for U-absolute coequalizer pairs.*

I will not explain what 'creates coequalizers for...' means; all that matters is that it can be verified without knowing anything about the left adjoint to U.

Example 7.15 The General Adjoint Functor Theorem shows that each of the functors (11) has a left adjoint (Lecture 3). Using the Monadicity Theorem, it can be shown that each is in fact monadic. So too are the forgetful functors

$$\begin{array}{ccc} \text{(topological groups)} & \mathbf{CptHff} & \mathbf{Cat} \\ \downarrow & \downarrow & \downarrow \\ \mathbf{Top}, & \mathbf{Set}, & \text{(directed graphs).} \end{array}$$

Proposition 7.16 *Any monadic functor*

a. is faithful,

b. reflects isomorphisms, and

c. creates limits.

The terminology in (b) and (c) is defined below.

Proof Omitted. It suffices to prove that for any monad T, the properties hold for U^T. $\square$

Definition 7.17 A functor $U : \mathcal{B} \longrightarrow \mathcal{A}$ **reflects isomorphisms** if for all maps g in $\mathcal{B}$,

$$Ug \text{ is an isomorphism } \Rightarrow g \text{ is an isomorphism.}$$

Example 7.18 U : **Gp** $\longrightarrow$ **Set** reflects isomorphisms, that is, any bijective homomorphism is an isomorphism. U : **CptHff** $\longrightarrow$ **Set** reflects isomorphisms, that is, any continuous bijection between compact Hausdorff spaces is a homeomorphism. U : **Top** $\longrightarrow$ **Set** does not reflect isomorphisms, so is not monadic.

Definition 7.19 A functor $U : \mathcal{B} \longrightarrow \mathcal{A}$ **creates limits strictly** if whenever $D : \mathbb{I} \longrightarrow \mathcal{B}$ is a diagram in $\mathcal{B}$ and $(L \xrightarrow{p_I} UDI)_{I \in \mathbb{I}}$ is a limit cone on $U \circ D$ in $\mathcal{A}$,

a. there is a unique lift of $(L \xrightarrow{p_I} UDI)_{I \in \mathbb{I}}$ to a cone on D in $\mathcal{B}$, and

b. this cone on D is a limit cone.

This definition is over-strict: 'unique' implicitly refers to equality of objects. The correct definition of creation replaces equality with isomorphism; it is a little harder to digest. But in fact, U^T does create limits in this strict sense, as do many other monadic functors.

Example 7.20 $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$ creates limits strictly. In the case of binary products, this says that for any two groups B_1 and B_2 there is a unique group structure on the set $B_1 \times B_2$ for which both projections $B_1 \times B_2 \longrightarrow B_i$ are homomorphisms, and that with this group structure,

$$B_1 \longleftarrow B_1 \times B_2 \longrightarrow B_2$$

is a product diagram in $\mathbf{Gp}$.

If $\mathcal{A}$ has all limits and $U : \mathcal{B} \longrightarrow \mathcal{A}$ creates limits then $\mathcal{B}$ has all limits too. So any category monadic over **Set** has all limits.

Example 7.21 **Field** has no terminal object: for what would its characteristic be? Hence **Field** is not monadic over **Set**. (In fact, U : **Field** $\longrightarrow$ **Set** does not even have a left adjoint.)

Much harder is the fact that any category monadic over **Set** has all *co*limits. This implies, for example, that it is possible to form the coproduct of groups, which is perhaps not instantly obvious.

Exercises

7.22 Complete the following sentences in the style of the remarks accompanying the definitions of category, functor, and natural transformation, and the Yoneda Lemma (1.3(a), after 1.11, after 2.2, and before 4.7):

> 'Loosely, a monad on $\mathcal{A}$ is ...'
> 'Loosely, a T-algebra is ...'

7.23 Find three more examples of non-monadic functors.

7.24 Consider the following passage:

> We are thus led to consider conjugacy idempotents on groups. There is a universal example, that is, there is a group F together with an endomorphism f and an element $\alpha \in F$ such that $f^2 = f^\alpha$ with the property that for any other such triple (G, g, β) there exists a unique $h : F \longrightarrow$

G such that

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d, "f"'] & G \arrow[d, "g"] \\ F \arrow[r, "h"] & G \end{tikzcd}$$

commutes and $h(\alpha) = \beta$.

(Peter Freyd, Alex Heller, Splitting homotopy idempotents II, *Journal of Pure and Applied Algebra* 89 (1993), 93–106. An **endomorphism** of an object A of a category is a map $A \longrightarrow A$. The notation f^2 means $f \circ f$, and f^α is defined by $f^\alpha(x) = \alpha^{-1} f(x) \alpha$.)

Why is there a universal example?