Lecture 1
Categories and functors

Definition 1.1 A category $\mathcal{A}$ consists of

- a collection $\text{ob}(\mathcal{A})$ (whose elements are called the objects of $\mathcal{A}$)
- for each $A, B \in \text{ob}(\mathcal{A})$, a collection $\mathcal{A}(A, B)$ (whose elements are called the maps or morphisms or arrows from $A$ to $B$)
- for each $A, B, C \in \text{ob}(\mathcal{A})$, a function

$$
\mathcal{A}(B, C) \times \mathcal{A}(A, B) \quad \longrightarrow \quad \mathcal{A}(A, C),
$$

$(g, f) \mapsto g \circ f$

(called composition)

- for each $A \in \text{ob}(\mathcal{A})$, an element $1_A \in \mathcal{A}(A, A)$ (the identity on $A$) satisfying

  - associativity: $(h \circ g) \circ f = h \circ (g \circ f)$ for all $A, B, C, D \in \text{ob}(\mathcal{A})$, $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$, and $h \in \mathcal{A}(C, D)$
  
- unit laws: $f \circ 1_A = f = 1_B \circ f$ for all $A, B \in \text{ob}(\mathcal{A})$ and $f \in \mathcal{A}(A, B)$.

Notation 1.2 We often write:

$$
A \in \mathcal{A} \quad \text{to mean} \quad A \in \text{ob}(\mathcal{A})
$$

$$
f : A \underset{f}{\longrightarrow} B \quad \text{or} \quad A \overset{f}{\longrightarrow} B \quad \text{to mean} \quad f \in \mathcal{A}(A, B)
$$

$$
gf \quad \text{to mean} \quad g \circ f.
$$

People often write $\mathcal{A}(A, B)$ as $\text{Hom}_\mathcal{A}(A, B)$ or $\text{Hom}(A, B)$.

Remarks 1.3 a. Loosely, a category is a system of objects and arrows in which any string of arrows

$$
A_0 \overset{f_1}{\longrightarrow} \cdots \overset{f_n}{\longrightarrow} A_n
$$
gives rise to precisely one arrow $A_0 \longrightarrow A_n$. When $n = 2$, this is composition; when $n = 0$, it is the formation of identities; when $n = 3$, the ‘precisely one’ part implies the associativity law; and when $n = 0$, it implies the unit laws.
b. I will say as little as you let me about set theory. It suffices to make a naive distinction between small and large collections, which can be interpreted as meaning ‘sets’ and ‘proper classes’ respectively. A category $\mathcal{A}$ is locally small if $\mathcal{A}(A, B)$ is a small collection for each $A$ and $B$. (Many authors build locally small into their definition of category.) A category $\mathcal{A}$ is small if it is locally small and the collection $\text{ob}(\mathcal{A})$ is small.

The most obvious examples of categories come under the banner ‘categories of mathematical structures’.

**Example 1.4** There is a category $\textbf{Set}$ in which the objects are sets and the maps are functions. Similarly:

- $\textbf{Top}$ is topological spaces and continuous maps
- $\textbf{Gp}$ is groups and homomorphisms
- $\textbf{Ab}$ is abelian groups and homomorphisms
- $k$-$\text{Mod}$ is (left) $k$-modules and homomorphisms, for any ring $k$.

**Example 1.5** There is an obvious notion of subcategory. For instance, there is a subcategory of $\textbf{Ab}$ consisting of all abelian groups with between 50 and 60 elements and all surjective homomorphisms between them.

A map $f : A \longrightarrow B$ in a category is an isomorphism if there exists $f' : B \longrightarrow A$ satisfying $f'f = 1_A$ and $ff' = 1_B$. There is at most one such $f'$ (exercise), so we may write $f' = f^{-1}$ and call it the inverse of $f$. We also call $A$ and $B$ isomorphic and write $A \cong B$.

The next examples are ‘categories as mathematical structures’.

**Example 1.6** A (partial) order on a set $A$ is a binary relation $\leq$ on $A$ that is reflexive, transitive, and antisymmetric ($a \leq b \leq a$ implies $a = b$).

Examples: $A = \mathbb{R}$ and $\leq$ has the usual meaning; $A$ is the set of subsets of some fixed set and $\subseteq$; $A = \mathbb{N}$ and $a \leq b$ means $a|b$.

An ordered set $(A, \leq)$ can be regarded as a category $\mathcal{A}$ in which each ‘hom-set’ $\mathcal{A}(a, b)$ has at most one element. The objects of $\mathcal{A}$ are the elements of $A$, and there is an arrow $a \longrightarrow b$ if and only if $a \leq b$. It doesn’t matter what you call this arrow; you can think of it as ‘the assertion that $a \leq b$’. This example shows that the ‘maps’ in a category need not be remotely like ‘maps’ in the sense of functions.
Example 1.7 A **monoid** is a set equipped with an associative binary operation and a two-sided unit (e.g. $(\mathbb{N}, +, 0)$).

A small category with precisely one object is the same thing as a monoid. For if the object is called $\star$, say, then such a category consists of a single hom-set $\mathcal{A}(\star, \star)$ together with an associative binary operation (composition) and a two-sided unit (the identity on $\star$).

Example 1.8 In particular, a group is the same thing as a one-object small category in which every arrow is an isomorphism.

Example 1.9 A **groupoid** is a category in which every arrow is an isomorphism. Every topological space $X$ has a **fundamental groupoid** $\Pi_1(X)$, whose objects are the points of $X$ and whose arrows $x \to y$ are the homotopy classes of paths from $x$ to $y$.

**Digression 1.10** You might have noticed that in many categories $\mathcal{A}$, the sets $\mathcal{A}(A, B)$ carry extra structure. For instance, if $\mathcal{A} = k\text{-Mod}$ then they are abelian groups, and if $\mathcal{A}$ is a suitable category of spaces then they carry a topology. Such things are called ‘enriched categories’.

Homological algebra works with ‘abelian categories’. An **Ab-category** is a category in which each $\mathcal{A}(A, B)$ has the structure of an abelian group and composition is bilinear. An **additive category** is an Ab-category satisfying further conditions. An **abelian category** is an additive category satisfying further conditions still, enabling one to define and manipulate exact sequences inside the category. The basic example is $k\text{-Mod}$ where $k$ is a commutative ring.

**Definition 1.11** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **functor** $F : \mathcal{A} \to \mathcal{B}$ consists of

- a function
  \[
  \begin{align*}
  \text{ob}(\mathcal{A}) & \longrightarrow \text{ob}(\mathcal{B}), \\
  A & \longmapsto FA
  \end{align*}
  \]

- for each $A, A' \in \text{ob}(\mathcal{A})$, a function
  \[
  \begin{align*}
  \mathcal{A}(A, A') & \longrightarrow \mathcal{B}(FA, FA'), \\
  f & \longmapsto Ff
  \end{align*}
  \]

such that

- $F(f' \circ f) = Ff' \circ Ff$ for all $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathcal{A}$
• $F1_A = 1_{FA}$ for all $A \in \mathcal{A}$.

Loosely, a functor $\mathcal{A} \to \mathcal{B}$ is something that associates to every object $A$ of $\mathcal{A}$ an object $FA$ of $\mathcal{B}$ and to every string of arrows

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n$$

in $\mathcal{A}$ precisely one arrow $FA_0 \to FA_n$. When $n = 1$ this says what the $(Ff)$s are; when $n = 2$ and $n = 0$ it implies that $F$ preserves composition and identities.

**Example 1.12** ‘Forgetful functors’ (an informal term) are functors that forget structure or properties. For instance, there is a functor $\textbf{Gp} \to \textbf{Set}$ sending every group to its underlying set; it ‘forgets’ the group structure and that homomorphisms are homomorphisms. There is a forgetful functor $\textbf{Ab} \to \textbf{Gp}$, which might also be called an inclusion; it forgets the property of being abelian.

**Example 1.13** In the other direction, ‘free functors’ add in structure or properties freely. For instance, there is a functor $\textbf{Set} \to \textbf{Gp}$ that forms the free group on each set, and a functor $F : \textbf{Gp} \to \textbf{Ab}$ that sends each group to its largest abelian quotient: $F(X)$ is $X^{ab} = X/[X,X]$, the abelianization of $X$.

**Example 1.14** Any monoid $M$ (e.g. a group) can be regarded as a one-object category (1.7). A functor $M \to \textbf{Set}$ is just a set with a left $A$-action. Similarly, a functor from $A \to \textbf{k-Mod}$ is a $k$-linear representation of $A$.

Some functors reverse the direction of arrows: an arrow $\xrightarrow{f} A' \to A$ in $\mathcal{A}$ gives rise to an arrow $\xleftarrow{Ff} FA' \to FA$ in $\mathcal{B}$. This can be made precise as follows. Given a category $\mathcal{A}$, the **opposite** or **dual** category $\mathcal{A}^\text{op}$ is defined by $\text{ob}(\mathcal{A}^\text{op}) = \text{ob}(\mathcal{A})$ and $\mathcal{A}^\text{op}(A', A) = \mathcal{A}(A, A')$; composition and identities are as in $\mathcal{A}$, but reversed. A functor $\mathcal{A}^\text{op} \to \mathcal{B}$ (or equivalently, $\mathcal{A} \to \mathcal{B}^\text{op}$) is called a **contravariant functor**; ordinary functors are sometimes called **covariant**, for emphasis.

**Example 1.15** Taking duals defines a functor

$$\text{Vect}_k^\text{op} \to \text{Vect}_k, \quad V \mapsto V^*,$$

where $\text{Vect}_k$ is the category of vector spaces over a field $k$. 10
Example 1.16 Homology defines a functor $H_* : \text{Top} \longrightarrow \text{GrAb} = (\text{graded abelian groups})$; cohomology defines a functor $H^* : \text{Top}^{\text{op}} \longrightarrow \text{GrAb}$.

Example 1.17 Fix a topological space $X$. The set $\text{Open}(X)$ of open subsets of $X$ can be ordered by inclusion, and so forms a category (1.6). A functor $\text{Open}(X)^{\text{op}} \longrightarrow \text{Set}$ is called a presheaf (of sets) on $X$. Concretely, a presheaf on $X$ consists of a set $F(U)$ for each open set $U$, and, for each pair $U' \subseteq U$ of open sets, a function $F(U) \longrightarrow F(U')$, satisfying some axioms. Example: $F(U)$ is the set of continuous maps $U \longrightarrow \mathbb{R}$, and the functions $F(U) \longrightarrow F(U')$ are given by restriction.

More generally, a presheaf on a category $\mathcal{A}$ is a functor $\mathcal{A}^{\text{op}} \longrightarrow \text{Set}$.

Functors are the structure-preserving maps of categories; they can be composed, so there is a (large) category $\text{Cat}$ consisting of small categories and functors. Informally, there is also a (huge) category $\text{CAT}$ consisting of all categories and functors.

In the next lecture we’ll see that there is a further notion of map between functors.

Exercises

1.18 There is a category $\text{Toph}$ whose objects are topological spaces and whose arrows $X \longrightarrow Y$ are homotopy classes of continuous maps from $X$ to $Y$. What would you need to know about homotopy in order to prove that this is a category? What does it mean for two objects of $\text{Toph}$ to be isomorphic?

1.19 Two categories $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, written $\mathcal{A} \cong \mathcal{B}$, if they are isomorphic as objects of $\text{CAT}$. Prove that any group, regarded as a one-object category, is isomorphic to its opposite. Find a monoid not isomorphic to its opposite.

1.20 Prove that functors preserve isomorphism.