Lecture 1

Categories and functors

Definition 1.1 A **category** $\mathcal{A}$ consists of

- a collection $\mathrm{ob}(\mathcal{A})$ (whose elements are called the **objects** of $\mathcal{A}$)
- for each $A, B \in \mathrm{ob}(\mathcal{A})$, a collection $\mathcal{A}(A, B)$ (whose elements are called the **maps** or **morphisms** or **arrows** from A to B)
- for each $A, B, C \in \mathrm{ob}(\mathcal{A})$, a function

$$\begin{array}{ccc} \mathcal{A}(B,C) \times \mathcal{A}(A,B) & \longrightarrow & \mathcal{A}(A,C), \\ (g,f) & \longmapsto & g \circ f \end{array}$$

  (called **composition**)

- for each $A \in \mathrm{ob}(\mathcal{A})$, an element $1_A \in \mathcal{A}(A, A)$ (the **identity** on A)

satisfying

- **associativity**: $(h \circ g) \circ f = h \circ (g \circ f)$ for all $A, B, C, D \in \mathrm{ob}(\mathcal{A})$, $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$, and $h \in \mathcal{A}(C, D)$
- **unit laws**: $f \circ 1_A = f = 1_B \circ f$ for all $A, B \in \mathrm{ob}(\mathcal{A})$ and $f \in \mathcal{A}(A, B)$.

Notation 1.2 We often write:

$$\begin{array}{rll} A \in \mathcal{A} & \text{to mean} & A \in \mathrm{ob}(\mathcal{A}) \\ f : A \longrightarrow B \text{ or } A \xrightarrow{f} B & \text{to mean} & f \in \mathcal{A}(A,B) \\ gf & \text{to mean} & g \circ f. \end{array}$$

People often write $\mathcal{A}(A, B)$ as $\mathrm{Hom}_{\mathcal{A}}(A, B)$ or $\mathrm{Hom}(A, B)$.

Remarks 1.3 a. Loosely, a category is a system of objects and arrows in which any string of arrows

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n$$

gives rise to precisely one arrow $A_0 \longrightarrow A_n$. When $n = 2$, this is composition; when $n = 0$, it is the formation of identities; when $n = 3$, the 'precisely one' part implies the associativity law; and when $n = 0$, it implies the unit laws.

b. I will say as little as you let me about set theory. It suffices to make a naive distinction between **small** and **large** collections, which can be interpreted as meaning 'sets' and 'proper classes' respectively. A category $\mathcal{A}$ is **locally small** if $\mathcal{A}(A, B)$ is a small collection for each A and B. (Many authors build locally small into their definition of category.) A category $\mathcal{A}$ is **small** if it is locally small and the collection $\mathrm{ob}(\mathcal{A})$ is small.

c. A harmless convention is that $\mathcal{A}(A, B) \cap \mathcal{A}(A', B') = \emptyset$ unless $A = A'$ and $B = B'$. If $f \in \mathcal{A}(A, B)$, the **domain** of f is A and the **codomain** of f is B.

The most obvious examples of categories come under the banner 'categories *of* mathematical structures'.

Example 1.4 There is a category **Set** in which the objects are sets and the maps are functions. Similarly:

- **Top** is topological spaces and continuous maps
- **Gp** is groups and homomorphisms
- **Ab** is abelian groups and homomorphisms
- k-**Mod** is (left) k-modules and homomorphisms, for any ring k.

Example 1.5 There is an obvious notion of **subcategory**. For instance, there is a subcategory of **Ab** consisting of all abelian groups with between 50 and 60 elements and all surjective homomorphisms between them.

A map $f : A \longrightarrow B$ in a category is an **isomorphism** if there exists $f' : B \longrightarrow A$ satisfying $f'f = 1_A$ and $ff' = 1_B$. There is at most one such f' (exercise), so we may write $f' = f^{-1}$ and call it the **inverse** of f. We also call A and B isomorphic and write $A \cong B$.

The next examples are 'categories *as* mathematical structures'.

Example 1.6 A **(partial) order** on a set A is a binary relation $\leq$ on A that is reflexive, transitive, and antisymmetric ($a \leq b \leq a$ implies $a = b$). Examples: $A = \mathbb{R}$ and $\leq$ has the usual meaning; A is the set of subsets of some fixed set and $\leq$ is $\subseteq$; $A = \mathbb{N}$ and $a \leq b$ means $a|b$.

An ordered set $(A, \leq)$ can be regarded as a category $\mathcal{A}$ in which each 'hom-set' $\mathcal{A}(a, b)$ has at most one element. The objects of $\mathcal{A}$ are the elements of A, and there is an arrow $a \longrightarrow b$ if and only if $a \leq b$. It doesn't matter what you call this arrow; you can think of it as 'the assertion that $a \leq b$'. This example shows that the 'maps' in a category need not be remotely like 'maps' in the sense of functions.

Example 1.7 A **monoid** is a set equipped with an associative binary operation and a two-sided unit (e.g. $(\mathbb{N}, +, 0)$).

A small category with precisely one object is the same thing as a monoid. For if the object is called $\star$, say, then such a category consists of a single hom-set $\mathcal{A}(\star, \star)$ together with an associative binary operation (composition) and a two-sided unit (the identity on $\star$).

Example 1.8 In particular, a group is the same thing as a one-object small category in which every arrow is an isomorphism.

Example 1.9 A **groupoid** is a category in which every arrow is an isomorphism. Every topological space X has a **fundamental groupoid** $\Pi_1(X)$, whose objects are the points of X and whose arrows $x \longrightarrow y$ are the homotopy classes of paths from x to y.

Digression 1.10 You might have noticed that in many categories $\mathcal{A}$, the sets $\mathcal{A}(A, B)$ carry extra structure. For instance, if $\mathcal{A} = k$-**Mod** then they are abelian groups, and if $\mathcal{A}$ is a suitable category of spaces then they carry a topology. Such things are called 'enriched categories'.

Homological algebra works with 'abelian categories'. An **Ab-category** is a category in which each $\mathcal{A}(A, B)$ has the structure of an abelian group and composition is bilinear. An **additive category** is an **Ab**-category satisfying further conditions. An **abelian category** is an additive category satisfying further conditions still, enabling one to define and manipulate exact sequences inside the category. The basic example is k-**Mod** where k is a commutative ring.

Definition 1.11 Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **functor** $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of

- a function
$$\begin{array}{ccc} \mathrm{ob}(\mathcal{A}) & \longrightarrow & \mathrm{ob}(\mathcal{B}), \\ A & \longmapsto & FA \end{array}$$
- for each $A, A' \in \mathrm{ob}(\mathcal{A})$, a function
$$\begin{array}{ccc} \mathcal{A}(A, A') & \longrightarrow & \mathcal{B}(FA, FA'), \\ f & \longmapsto & Ff \end{array}$$

such that

- $F(f' \circ f) = Ff' \circ Ff$ for all $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathcal{A}$

- $F1_A = 1_{FA}$ for all $A \in \mathcal{A}$.

Loosely, a functor $\mathcal{A} \longrightarrow \mathcal{B}$ is something that associates to every object A of $\mathcal{A}$ an object FA of $\mathcal{B}$ and to every string of arrows

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n$$

in $\mathcal{A}$ precisely one arrow $FA_0 \longrightarrow FA_n$. When $n = 1$ this says what the (Ff)s are; when $n = 2$ and $n = 0$ it implies that F preserves composition and identities.

Example 1.12 'Forgetful functors' (an informal term) are functors that forget structure or properties. For instance, there is a functor **Gp** $\longrightarrow$ **Set** sending every group to its underlying set; it 'forgets' the group structure and that homomorphisms are homomorphisms. There is a forgetful functor **Ab** $\longrightarrow$ **Gp**, which might also be called an inclusion; it forgets the property of being abelian.

Example 1.13 In the other direction, 'free functors' add in structure or properties freely. For instance, there is a functor **Set** $\longrightarrow$ **Gp** that forms the free group on each set, and a functor F : **Gp** $\longrightarrow$ **Ab** that sends each group to its largest abelian quotient: $F(X)$ is $X^{\mathrm{ab}} = X/[X, X]$, the **abelianization** of X.

Example 1.14 Any monoid M (e.g. a group) can be regarded as a one-object category (1.7). A functor $M \longrightarrow$ **Set** is just a set with a left A-action. Similarly, a functor from $A \longrightarrow k$-**Mod** is a k-linear representation of A.

Some functors reverse the direction of arrows: an arrow $A \xrightarrow{f} A'$ in $\mathcal{A}$ gives rise to an arrow $FA \xleftarrow{Ff} FA'$ in $\mathcal{B}$. This can be made precise as follows. Given a category $\mathcal{A}$, the **opposite** or **dual** category $\mathcal{A}^{\mathrm{op}}$ is defined by $\mathrm{ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{A})$ and $\mathcal{A}^{\mathrm{op}}(A', A) = \mathcal{A}(A, A')$; composition and identities are as in $\mathcal{A}$, but reversed. A functor $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$ (or equivalently, $\mathcal{A} \longrightarrow \mathcal{B}^{\mathrm{op}}$) is called a **contravariant functor**; ordinary functors are sometimes called **covariant**, for emphasis.

Example 1.15 Taking duals defines a functor

$$\begin{array}{ccc} \mathbf{Vect}_k^{\mathrm{op}} & \longrightarrow & \mathbf{Vect}_k, \\ V & \longmapsto & V^*, \end{array}$$

where $\mathbf{Vect}_k$ is the category of vector spaces over a field k.

Example 1.16 Homology defines a functor $H_* : \mathbf{Top} \longrightarrow \mathbf{GrAb} =$ (graded abelian groups); cohomology defines a functor $H^* : \mathbf{Top}^{\mathrm{op}} \longrightarrow \mathbf{GrAb}$.

Example 1.17 Fix a topological space X. The set $\mathbf{Open}(X)$ of open subsets of X can be ordered by inclusion, and so forms a category (1.6). A functor $\mathbf{Open}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is called a **presheaf** (of sets) on X. Concretely, a presheaf on X consists of a set $F(U)$ for each open set U, and, for each pair $U' \subseteq U$ of open sets, a function $F(U) \longrightarrow F(U')$, satisfying some axioms. Example: $F(U)$ is the set of continuous maps $U \longrightarrow \mathbb{R}$, and the functions $F(U) \longrightarrow F(U')$ are given by restriction.

More generally, a **presheaf** on a category $\mathcal{A}$ is a functor $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

Functors are the structure-preserving maps of categories; they can be composed, so there is a (large) category **Cat** consisting of small categories and functors. Informally, there is also a (huge) category **CAT** consisting of all categories and functors.

In the next lecture we'll see that there is a further notion of map between functors.

Exercises

1.18 There is a category **Toph** whose objects are topological spaces and whose arrows $X \longrightarrow Y$ are homotopy classes of continuous maps from X to Y. What would you need to know about homotopy in order to prove that this is a category? What does it mean for two objects of **Toph** to be isomorphic?

1.19 Two categories $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic**, written $\mathcal{A} \cong \mathcal{B}$, if they are isomorphic as objects of **CAT**. Prove that any group, regarded as a one-object category, is isomorphic to its opposite. Find a monoid not isomorphic to its opposite.

1.20 Prove that functors preserve isomorphism.