Lecture 4

Representability

Hold on to your seat.

Construction 4.1 Fix a locally small category $\mathcal{A}$.

(a) Fix an object B of $\mathcal{A}$.

- For each $A \in \mathcal{A}$ there is a set $\mathcal{A}(A, B)$.
- For each $(A' \xrightarrow{f} A)$ in $\mathcal{A}$ there is a function

  $$\begin{array}{rccc} f^* = - \circ f : & \mathcal{A}(A, B) & \longrightarrow & \mathcal{A}(A', B), \\ & p & \longmapsto & p \circ f. \end{array}$$

  (Note the change of direction, and compare dual vector spaces: $\mathcal{A} = \mathbf{Vect}_k$, $B = k$.)

So there is a functor

$$H_B = \mathcal{A}(-, B) : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

$$\begin{tikzcd} A \arrow[r, mapsto] & \mathcal{A}(A, B) \arrow[d, "f^*"] \\ A' \arrow[u, "f"] \arrow[r, mapsto] & \mathcal{A}(A', B). \end{tikzcd}$$

with $f \longmapsto f^*$.

(a′) Fix an object A of $\mathcal{A}$.

- For each $B \in \mathcal{A}$ there is a set $\mathcal{A}(A, B)$.
- For each $(B \xrightarrow{g} B')$ in $\mathcal{A}$ there is a function

$$\begin{array}{rccc} g_* = g \circ - : & \mathcal{A}(A,B) & \longrightarrow & \mathcal{A}(A,B'), \\ & p & \longmapsto & g \circ p. \end{array}$$

So there is a functor

$$\begin{array}{rcl} H^A = \mathcal{A}(A, -) : \; \mathcal{A} & \longrightarrow & \mathbf{Set} \end{array}$$

$$\begin{tikzcd} B \arrow[d, "g"] & \longmapsto & \mathcal{A}(A,B) \arrow[d, "g_*"'] \\ B' & \longmapsto & \mathcal{A}(A,B'). \end{tikzcd}$$

(b) Fix an arrow $(B \xrightarrow{g} B')$ in $\mathcal{A}$. There is a natural transformation

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[r, bend left=50, "H_B"{name=U}] \arrow[r, bend right=50, "H_{B'}"'{name=D}] & \mathbf{Set} \arrow[from=U, to=D, Rightarrow, "H_g"] \end{tikzcd}$$

whose component at $A \in \mathcal{A}$ is

$$H_B(A) = \mathcal{A}(A, B) \xrightarrow{g_*} \mathcal{A}(A, B') = H_{B'}(A).$$

(b′) Fix an arrow $(A' \xrightarrow{f} A)$ in $\mathcal{A}$. There is a natural transformation

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "H^A"{name=U}] \arrow[r, bend right=50, "H^{A'}"'{name=D}] & \mathbf{Set} \arrow[Rightarrow, from=U, to=D, "H^f"] \end{tikzcd}$$

(note the change of direction) whose component at $B \in \mathcal{A}$ is

$$H^A(B) = \mathcal{A}(A, B) \xrightarrow{f^*} \mathcal{A}(A', B) = H^{A'}(B).$$

(c) Putting (a) and (b) together gives a functor

$$\begin{array}{rccc} H_\bullet : & \mathcal{A} & \longrightarrow & [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \\ & B & \longmapsto & H_B \\ & g \big\downarrow & \longmapsto & \big\downarrow H_g \\ & B' & \longmapsto & H_{B'}, \end{array}$$

the (covariant) **Yoneda embedding** of $\mathcal{A}$.

(c′) Putting (a′) and (b′) together gives a functor

$$
\begin{array}{rcl}
H^{\bullet}\colon \ \mathcal{A}^{\mathrm{op}} & \longrightarrow & [\mathcal{A}, \mathbf{Set}] \\
A & \longmapsto & H^{A} \\
\Big\uparrow f & \longmapsto & H^{f} \Big\downarrow \\
A' & \longmapsto & H^{A'},
\end{array}
$$

the contravariant Yoneda embedding of $\mathcal{A}$. (Every concept and theorem about arbitrary categories has a **dual** concept or theorem, obtained by reversing all the arrows in the categories—or formally, taking opposite categories. The contravariant Yoneda embedding is the dual concept of the covariant Yoneda embedding.)

Summary: given a locally small category $\mathcal{A}$,

(a), (a′)	$A \in \mathcal{A}$	gives	$\mathcal{A}^{\mathrm{op}} \xrightarrow{H_A} \mathbf{Set}$,	$\mathcal{A} \xrightarrow{H^A} \mathbf{Set}$
(b), (b′)	$(A' \xrightarrow{f} A)$ in $\mathcal{A}$	gives	$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[r, bend left, "H_{A'}"] \arrow[r, bend right, "H_A"'] \arrow[r, phantom, "\Downarrow H_f"] & \mathbf{Set} \end{tikzcd}$,	$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "H^{A'}"] \arrow[r, bend right, "H^A"'] \arrow[r, phantom, "\Uparrow H^f"] & \mathbf{Set} \end{tikzcd}$
(c), (c′)	we have		$\mathcal{A} \xrightarrow{H_\bullet} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$,	$\mathcal{A}^{\mathrm{op}} \xrightarrow{H^\bullet} [\mathcal{A}, \mathbf{Set}]$.

Definition 4.2 Let $\mathcal{A}$ be a locally small category. A functor $X : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is **representable** if $X \cong H_A$ for some $A \in \mathcal{A}$, and a **representation** of X is a pair (A, α) where $A \in \mathcal{A}$ and α is a natural isomorphism $H_A \longrightarrow X$. Dual definitions apply to covariant functors, $X : \mathcal{A} \longrightarrow \mathbf{Set}$.

Proposition 4.3 *Let $U : \mathcal{A} \longrightarrow \mathbf{Set}$. If U has a left adjoint then U is representable.*

Proof Take $F \dashv U$ and write 1 for a one-element set. Then

$$UA \cong \mathbf{Set}(1, UA) \cong \mathcal{A}(F1, A)$$

naturally in $A \in \mathcal{A}$, so $U \cong H^{F1}$. $\square$

Example 4.4 The forgetful functor $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$ has a left adjoint F, so is representable. We have $U \cong H^{F1}$; but $F1$ is the free group on one generator, $\mathbb{Z}$, so $U \cong H^{\mathbb{Z}}$. So $U(A) \cong H^{\mathbb{Z}}(A) = \mathbf{Gp}(\mathbb{Z}, A)$ for all groups A (and naturally in A). This says that an element of a group A is the same thing as a homomorphism $\mathbb{Z} \longrightarrow A$.

Example 4.5 Similarly, $U : \mathbf{Ring} \longrightarrow \mathbf{Set}$ is represented by the free ring $\mathbb{Z}[x]$ on one generator and $U : k\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$ by the free k-module k on one generator.

Example 4.6 Fix k-modules C and D. There is a functor

$$\begin{array}{rccl} \mathbf{Bilin}(C, D; -) : & k\text{-}\mathbf{Mod} & \longrightarrow & \mathbf{Set}, \\ & E & \longmapsto & \mathbf{Bilin}(C, D; E) \\ & & & = \{\text{bilinear maps } C \times D \longrightarrow E\}. \end{array}$$

We have

$$\mathbf{Bilin}(C, D; E) \cong k\text{-}\mathbf{Mod}(C \otimes D, E)$$

naturally in E, so the functor $\mathbf{Bilin}(C, D; -)$ is representable.

*   *   *

We come now to a fundamental result: the Yoneda Lemma. Loosely, this says that an object $A \in \mathcal{A}$ and a functor $X : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ give rise to precisely one set. What can this mean?

- Such an A and X certainly give rise to the set $X(A)$.

- But $A \in \mathcal{A}$ induces $H_A : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, so A and X also give rise to the set $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$ of natural transformations of the form

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Set}. \end{tikzcd} \tag{4}$$

Yoneda says that these two sets are isomorphic in a canonical way. In other words, a natural transformation $H_A \longrightarrow X$ is the same thing as an element of $X(A)$.

Theorem 4.7 (Yoneda Lemma) *Let $\mathcal{A}$ be a locally small category. Then there is a bijection*

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \cong X(A)$$

natural in $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

Sketch proof Fix A and X. Given $\alpha : H_A \longrightarrow X$, we have $\alpha_A : H_A(A) = \mathcal{A}(A, A) \longrightarrow X(A)$, hence $\alpha_A(1_A) \in X(A)$. This gives a function

$$\begin{array}{ccc} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) & \longrightarrow & X(A) \\ \alpha & \longmapsto & \alpha_A(1_A) \end{array}$$

whose inverse can be constructed. Then check naturality in A and X. $\square$

There are three important corollaries.

Corollary 4.8 (Representation = universal element) *Let $\mathcal{A}$ be a locally small category and $X : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. Then representations of X are in one-to-one correspondence with pairs (A, u) where $A \in \mathcal{A}$, $u \in X(A)$, and*

for any $B \in \mathcal{A}$ and $x \in X(B)$, there is a unique map $f : B \longrightarrow A$ satisfying $(Xf)(u) = x$. (5)

Think of u as a 'universal' or 'generic' element of X.

Sketch proof A representation of X is, by definition, a pair (A, α) consisting of an object $A \in \mathcal{A}$ and a natural isomorphism $\alpha : H_A \xrightarrow{\sim} X$. Yoneda says that natural transformations $H_A \longrightarrow X$ are the same as elements of $X(A)$, so we just have to check that α is invertible if and only if the corresponding element $u = \alpha_A(1_A)$ of $X(A)$ satisfies (5). $\square$

Example 4.9 Fix a commutative ring k and k-modules C and D, and consider again the functor

$$\mathbf{Bilin}(C, D; -) : k\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}.$$

A representation of $\mathbf{Bilin}(C, D; -)$ can be described in two equivalent ways:

a. as a module T together with a natural isomorphism

$$H^T = k\text{-}\mathbf{Mod}(T, -) \xrightarrow{\sim} \mathbf{Bilin}(C, D; -)$$

b. as a module T together with a bilinear map $u : C \times D \longrightarrow T$ such that

for any module B and bilinear map $x : C \times D \longrightarrow B$, there is a unique linear map $f : T \longrightarrow B$ making

$$\begin{tikzcd}
C \times D \arrow[r, "u"] \arrow[dr, "x"'] & T \arrow[d, "f"] \\
& B
\end{tikzcd}$$

commute.

Part (a) is just the definition of representation. Part (b) is the description given in the corollary (or rather its dual, concerning *co*variant functors).

It might look as if (b) says more than (a): that $\mathbf{Bilin}(C, D; -)$ and k-$\mathbf{Mod}(T, -)$ are not merely isomorphic, but isomorphic in a specific manner (by composition with u). This is an illusion; the word 'natural' in (a) carries a lot of weight.

The second corollary of the Yoneda Lemma is:

Corollary 4.10 (Yoneda embedding) *The Yoneda embedding is full and faithful.*

Sketch proof Let $A, B \in \mathcal{A}$ and take $X = H_B$ in the Yoneda Lemma: then a map $H_A \longrightarrow H_B$ is the same as an element of $H_B(A) = \mathcal{A}(A, B)$. □

This means that we can regard $\mathcal{A}$ as a full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, identifying $A \in \mathcal{A}$ with the corresponding representable H_A. Later we'll see why this is useful, and how every functor $X : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ can be built out of representables in roughly the same way that every number is built out of primes.

The third and final corollary is:

Corollary 4.11 (Uniqueness of representation) *For objects A and B of a locally small category $\mathcal{A}$,*

$$H_A \cong H_B \iff A \cong B \iff H^A \cong H^B.$$

Sketch proof By duality, it suffices to prove the first '$\iff$'.

'$\Rightarrow$' follows from the fact that $H_\bullet$ (like any functor) preserves isomorphism: Exercise 1.20.

'$\Leftarrow$' follows from $H_\bullet$ being full and faithful. □

The substance of this result is that $H_A \cong H_B \Rightarrow A \cong B$. Think of $H_A(C) = \mathcal{A}(C, A)$ as 'A viewed from C': then the result is that two objects are the same if they look the same from all viewpoints.

The category of sets is very unusual in this respect: for sets A and B,

$$A \cong B \iff H_A(1) \cong H_B(1),$$

so the implication $H_A \cong H_B \Rightarrow A \cong B$ is trivial when $\mathcal{A} = \mathbf{Set}$. In **Set**, it's enough to look at everything from the one-element set—all that matters about a set is its elements.

The category of groups is much more typical. Let A and B be groups. Then:

- $H_A(1) \cong H_B(1)$ no matter what A and B are
- $H_A(\mathbb{Z}) \cong H_B(\mathbb{Z})$ if and only if A and B have the same cardinality (but perhaps quite different group structures)
- $H_A(\mathbb{Z}/p\mathbb{Z}) \cong H_B(\mathbb{Z}/p\mathbb{Z})$ (for a prime p) if and only if A and B have the same number of elements of order p.

Each of these only gives partial information about the similarity of A and B, but a whole natural isomorphism $H_A \cong H_B$ tells us that $A \cong B$.

Example 4.12 Adjoints are unique. Suppose that $\mathcal{A} \underset{G}{\overset{F,F'}{\rightleftarrows}} \mathcal{B}$ with $F \dashv G$ and $F' \dashv G$. Then for each $A \in \mathcal{A}$ we have

$$\mathcal{B}(FA, B) \cong \mathcal{A}(A, GB) \cong \mathcal{B}(F'A, B)$$

naturally in B: hence $H^{FA} \cong H^{F'A}$. So $FA \cong F'A$ for all $A \in \mathcal{A}$. In fact this isomorphism is natural in A, that is, $F \cong F'$.

Exercises

4.13 Let $\mathcal{A}$ be a locally small category and $A, B \in \mathcal{A}$. Prove with your bare hands—no Yoneda Lemma—that if $H_A \cong H_B$ then $A \cong B$.

4.14 Understand the Yoneda Lemma.