Lecture 5

Limits

Example 5.1 Let A_1 and A_2 be objects of a category $\mathcal{A}$. A **product** of A_1 and A_2 is a triple (P, p_1, p_2) where

$$\begin{tikzcd} & A_1 \\ P \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & A_2, \end{tikzcd}$$

and the following universal property holds:

$$\begin{tikzcd} & & A_1 \\ B \arrow[urr, "s_1"] \arrow[drr, "s_2"'] \arrow[r, dotted, "\exists!"] & P \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\ & & A_2. \end{tikzcd}$$

That is, given any $B \in \mathcal{A}$ and $s_i \in \mathcal{A}(B, A_i)$ ($i = 1, 2$), there is a unique $(s_1, s_2) \in \mathcal{A}(B, P)$ such that $p_i \circ (s_1, s_2) = s_i$ for $i = 1, 2$.

Two objects A_1, A_2 may or may not have a product, but if they do then it is unique. More precisely, if (P, p_1, p_2) and (P', p'_1, p'_2) are both products of A_1 and A_2 then there is a unique isomorphism $j : P \longrightarrow P'$ such that $p'_i \circ j = p_i$ ($i = 1, 2$). We write $P = A_1 \times A_2$.

- $\mathcal{A} = \mathbf{Set}$: every pair (A_1, A_2) of sets has a product, the usual cartesian product $A_1 \times A_2$ with $p_i = (i\text{th projection})$. Then $(s_1, s_2)(b) = (s_1(b), s_2(b))$.

- $\mathcal{A} = \mathbf{Top}$: similarly, using the 'product topology'.

- $\mathcal{A} = k\text{-}\mathbf{Mod}$ (k a commutative ring): $A_1 \times A_2 = A_1 \oplus A_2$, with $p_i(a_1 + a_2) = a_i$.

- $\mathcal{A} = \mathbf{Gp}$: $\times$ is direct product.

Dually, a **coproduct** of A_1 and A_2 is a diagram

$$\begin{tikzcd} A_1 \arrow[dr, "q_1"] & \\ & C \\ A_2 \arrow[ur, "q_2"'] & \end{tikzcd}$$

with the dual universal property. We write $C = A_1 + A_2$ or $C = A_1 \amalg A_2$.

- $\mathcal{A} = \mathbf{Set}$: $\amalg$ is disjoint union and q_i is the inclusion of A_i into $A_1 \amalg A_2$.
- $\mathcal{A} = \mathbf{Top}$: similarly. So A is connected $\iff$ $A \ncong A_1 \amalg A_2$ for any nonempty A_1, A_2.
- $\mathcal{A} = k$-$\mathbf{Mod}$: $A_1 \oplus A_2$ is also the *co*product, with $q_i(a_i) = a_i$.
- $\mathcal{A} = \mathbf{Gp}$: coproduct is 'free product'.
- $\mathcal{A} = \mathbf{CommRing}$: coproduct is $\otimes$, with $q_1(a_1) = a_1 \otimes 1$ and similarly q_2.

Example 5.2 More generally, let $(A_i)_{i\in I}$ be a family of objects of a category $\mathcal{A}$, indexed over some set I. A **product** of $(A_i)_{i\in I}$ is an object $P \in \mathcal{A}$ together with a family $(p_i : P \longrightarrow A_i)_{i\in I}$ of maps, satisfying a universal property: given any family $(s_i : B \longrightarrow A_i)_{i\in I}$, there is a unique map $(s_i)_{i\in I} : B \longrightarrow P$ such that $p_j \circ (s_i)_{i\in I} = s_j$ for all $j \in I$. We write $P = \prod_{i\in I} A_i$. **Coproducts** are defined dually and written $\coprod_{i\in I} A_i$.

- $\mathcal{A} = \mathbf{Set}$: products are products, and coproducts are disjoint unions.

- $\mathcal{A} = k$-**Mod**: the product and coproduct of a *finite* family $(A_i)_{i\in I}$ are the same: $\bigoplus_{i\in I} A_i$. But in general, products and coproducts are different:

$$\begin{aligned}
\prod_{i\in I} A_i &= \{\text{families } (a_i)_{i\in I} \text{ with } a_i \in A_i\} \\
\coprod_{i\in I} A_i &= \{\text{families } (a_i)_{i\in I} \text{ with } a_i \in A_i \text{ and} \\
&\qquad a_i = 0 \text{ for all but finitely many } i \in I\}.
\end{aligned}$$

Example 5.3 An object P of a category $\mathcal{A}$ is **terminal** if for all $A \in \mathcal{A}$ there is precisely one map $A \longrightarrow$

P. Dually, an object $C \in \mathcal{A}$ is **initial** if for all $A \in \mathcal{A}$ there is precisely one map $C \longrightarrow A$. As in all these examples, terminal and initial objects are unique up to isomorphism, when they exist. (Exercise: prove it.)

A terminal object is just a product of the empty family $(A_i)_{i \in \emptyset}$; dually, an initial object is a coproduct of the empty family. So we write terminal objects as 1 and initial objects as 0.

- $\mathcal{A} = \mathbf{Set}$: $0 = \emptyset$ and $1 = \{\star\}$. (An n-element set is sometimes written as just n; then $+$ and $\times$ are what you think.)
- $\mathcal{A} = k$-**Mod**: $\{0\}$ is both initial and terminal.
- $\mathcal{A} = \mathbf{Ring}$: the initial object is $\mathbb{Z}$ and the terminal object is $\{0\}$.

Example 5.4 Take maps $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in a category $\mathcal{A}$. An **equalizer** of f and g is a map $E \xrightarrow{p} A$ satisfying $fp = gp$ and universal as such: for any $Q \xrightarrow{s} A$ satisfying $fs = gs$, there is a unique map $\overline{s} : Q \longrightarrow$

E such that $s = p\overline{s}$. Diagram:

$$\begin{tikzcd} Q \arrow[dr, dotted, "\exists!\overline{s}"'] \arrow[drr, "s"] & & & \\ & E \arrow[r, "p"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B. \end{tikzcd}$$

- $\mathcal{A} = \mathbf{Set}$: the equalizer is the subset on which f and g are equal. That is, $E = \{a \in A \mid f(a) = g(a)\}$ and p is inclusion.

- $\mathcal{A} = k$-**Mod**: the same holds; but then $E = \ker(f - g)$, so equalizers look like

  $$\ker(f-g) \hookrightarrow A \underset{g}{\overset{f}{\rightrightarrows}} B.$$

  In particular, $\ker(h : A \longrightarrow B)$ is the equalizer of $A \underset{0}{\overset{h}{\rightrightarrows}} B$.

Coequalizers are defined dually.

- $\mathcal{A} = \mathbf{Set}$: coequalizers look like

  $$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{q} B/\sim$$

where $\sim$ is the smallest equivalence relation on B satisfying $f(a) \sim g(a)$ for all $a \in A$, and q is the quotient map. (Compare Example 0.9.)

- $\mathcal{A} = k$-**Mod**: coequalizers look like

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{\text{quotient}} \operatorname{coker}(f-g) = B/\operatorname{im}(f-g).$$

Example 5.5 A **pullback** of a diagram

$$\begin{tikzcd}
A_1 \arrow[dr, "f_1"] & \\
& B \\
A_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

in a category $\mathcal{A}$ is a commutative square

$$\begin{tikzcd}
 & A_1 \arrow[dr, "f_1"] & \\
P \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & B \\
 & A_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

with the following universal property:

$$\begin{tikzcd}
 & & A_1 \arrow[dr, "f_1"] & \\
C \arrow[r, dotted, "\exists!"] \arrow[urr, "s_1"] \arrow[drr, "s_2"'] & P \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & & B. \\
 & & A_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

- $\mathcal{A} = \mathbf{Set}$: $P = \{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) =$

$f_2(a_2)\}$ and $p_i(a_1, a_2) = a_i$. Special cases:

$$\begin{tikzcd}
& A \arrow[dr, "f"] & \\
f^{-1}B' \arrow[ur, hook] \arrow[dr, "f|_{f^{-1}B'}"'] & & B \\
& B' \arrow[ur, hook] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& B_1 \arrow[dr, hook] & \\
B_1 \cap B_2 \arrow[ur, hook] \arrow[dr, hook] & & B \\
& B_2 \arrow[ur, hook] &
\end{tikzcd}$$

are pullback squares.

Pushouts are defined dually.

- $\mathcal{A} = \mathbf{Set}$: the pushout of

$$\begin{tikzcd}
& B_1 \\
A \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\
& B_2
\end{tikzcd}
\quad \text{is} \quad
\begin{tikzcd}
& B_1 \arrow[dr, "q_1"] & \\
A \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & (B_1 \amalg B_2)/\sim \\
& B_2 \arrow[ur, "q_2"'] &
\end{tikzcd}$$

where $\sim$ is the equivalence relation on $B_1 \amalg B_2$ generated by $f_1(a) \sim f_2(a)$ for all $a \in A$. Think of the pushout as 'B_1 and B_2 glued along A'. Special case: given subsets B_1, B_2 of some larger set,

$$\begin{tikzcd} & B_1 \arrow[dr, hook] & \\ B_1 \cap B_2 \arrow[ur, hook] \arrow[dr, hook] & & B_1 \cup B_2 \\ & B_2 \arrow[ur, hook] & \end{tikzcd}$$

is a pushout square.

Example 5.6 An arrow $A \xrightarrow{f} B$ in a category is **monic** if for every pair $Z \underset{a_2}{\overset{a_1}{\rightrightarrows}} A$ of maps into A, we have $fa_1 = fa_2 \Rightarrow a_1 = a_2$. Dually, f is **epic** if for all $B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C$ we have $g_1 f = g_2 f \Rightarrow g_1 = g_2$.

- $\mathcal{A} = \mathbf{Set}$: monic $\iff$ injective and epic $\iff$ surjective. Both $\Leftarrow$s are easy; for $\Rightarrow$, consider $Z = 1$ and $C = 2$.

- $\mathcal{A} = k\text{-}\mathbf{Mod}, \mathbf{Gp}, \mathbf{Ring}, \mathbf{LieAlg}, \ldots$: monic $\iff$ injective. Again $\Leftarrow$ is easy; for $\Rightarrow$, take

Z to be the free algebra on one generator. Also surjective $\Rightarrow$ epic. But in some of these categories there are epics that are not surjective: e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic in **Ring**.

The reason for including monics and epics in this list of examples is the following result.

Proposition 5.7 *A map $A \xrightarrow{f} B$ is monic if and only if the square*

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
A \arrow[ur, "1"] \arrow[dr, "1"'] & & B \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}$$

is a pullback, and dually for epics and pushouts (reversing the arrows).

Proof Exercise. $\square$

Here is the general definition. Notation: *small* categories will usually be in the typeface $\mathbb{A}, \mathbb{B}, \ldots$.

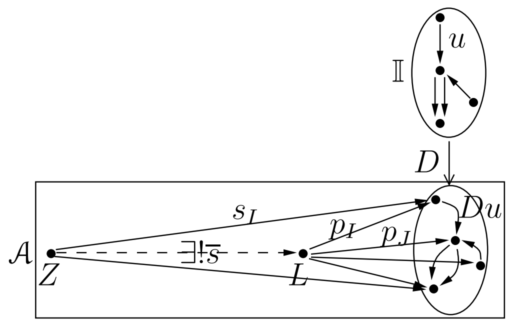

Figure 5-A: The limit $L = \lim_{\leftarrow \mathbb{I}} D$ of a diagram D, and another cone (Z, s) (with two legs not drawn)

Definition 5.8 (Figure 5-A) Let $\mathbb{I}$ be a small category and $\mathcal{A}$ any category.

a. A **diagram** over $\mathbb{I}$ in $\mathcal{A}$ is a functor $D : \mathbb{I} \longrightarrow \mathcal{A}$.

b. Let $Z \in \mathcal{A}$. A **cone on** D **with vertex** Z is a family $(Z \xrightarrow{s_I} DI)_{I \in \mathbb{I}}$ of maps such that $(Du) \circ s_I = s_J$ for all maps $I \xrightarrow{u} J$ in $\mathbb{I}$.

c. A cone $(L \xrightarrow{p_I} DI)_{I \in \mathbb{I}}$ is a **limit cone** if for any cone $(Z \xrightarrow{s_I} DI)_{I \in \mathbb{I}}$ there is a unique map

$\overline{s} : Z \longrightarrow L$ satisfying $p_I \circ \overline{s} = s_I$ for all $I \in \mathbb{I}$. Then L is the **limit** of D, written $L = \lim_{\leftarrow \mathbb{I}} D$ or $\lim_{\leftarrow I} DI$.

Cocones on a diagram and **colimits** are defined dually; the colimit of $D : \mathbb{I} \longrightarrow \mathcal{A}$ is written $\lim_{\rightarrow \mathbb{I}} D$ or $\lim_{\rightarrow I} DI$.

The (co)limit of a diagram may or may not exist, but if it does then it is unique up to canonical isomorphism.

All the examples above are (co)limits:

- A **discrete** category is one in which the only arrows are identities: so small discrete categories are just sets. A diagram in $\mathcal{A}$ over a discrete category $\mathbb{I}$ is just an indexed family $(A_I)_{I \in \mathbb{I}}$ of objects of $\mathcal{A}$. A (co)limit of such a diagram is just a (co)product.

- (Co)equalizers are (co)limits over the category $\mathbb{I} = (\bullet \rightrightarrows \bullet)$ with two objects and two non-identity arrows.

- Pullbacks are limits over the category

$$
\begin{tikzcd}
\bullet \arrow[dr] & \\
& \bullet \\
\bullet \arrow[ur] &
\end{tikzcd}
$$

  with three objects and two non-identity arrows; pushouts are colimits over the opposite of this category.

Example 5.9 Set has all limits (strictly, all *small* limits), that is, every diagram in **Set** has a limit. If $D : \mathbb{I} \longrightarrow \mathbf{Set}$ then

$$
\begin{aligned}
\lim_{\leftarrow \mathbb{I}} D &= \{(x_I)_{I\in\mathbb{I}} \in \prod DI \mid (Du)(x_I) = x_J \text{ for each } I \xrightarrow{u} J \text{ in } \mathbb{I}\}, \\
\lim_{\rightarrow \mathbb{I}} D &= \left(\coprod_{I\in\mathbb{I}} DI\right) / \sim
\end{aligned}
$$

where $\sim$ is the smallest equivalence relation such that $x \sim (Du)(x)$ whenever $I \xrightarrow{u} J$ and $x \in DI$. The limit projection $p_I : \lim_{\leftarrow\mathbb{I}} D \longrightarrow DI$ is ith projection, and the colimit coprojection $q_I : DI \longrightarrow \lim_{\rightarrow\mathbb{I}} DI$ sends $x \in DI$ to its equivalence class.

Example 5.10 Limits and colimits in **Top** are given by the same formulas, equipping the result with the smallest/largest topology for which the projections/coprojections are continuous.

Example 5.11 Limits in $\mathbf{Gp}, k\text{-}\mathbf{Mod}, \mathbf{Ring}, \ldots$ are computed as in **Set**, equipping the limit set with the unique algebraic structure for which the projections are homomorphisms. Such categories do have all colimits, but they are different from those in **Set**. See next lecture and 'Monads'.

Example 5.12 A category with all products and equalizers has all limits.

Example 5.13 Historically, the following case is important. An ordered set I is **directed** if every finite subset has an upper bound; equivalently, $I \neq \emptyset$ and $\forall i, j \in I,\ \exists k \in I:\ i, j \leq k$. A diagram $I \longrightarrow \mathcal{A}$ is then called a **direct system**, and a colimit of a direct system a **direct limit** (*sic*). In general, colimits do not commute with limits: e.g.

$$(A_1 + A_2) \times (B_1 + B_2) \ncong (A_1 \times B_1) + (A_2 \times B_2)$$

for sets A_i, B_i. However, direct limits commute with finite limits (= limits over a category $\mathbb{I}$ whose set of arrows is finite) in categories such as $\mathbf{Set}, \mathbf{Gp}, k\text{-}\mathbf{Mod}, \mathbf{Ring}, \ldots$.

Dually, if I is directed then a diagram $I^{\mathrm{op}} \longrightarrow \mathcal{A}$ is called an **inverse system**, and its limit an **inverse limit**. For example, $(\mathbb{N}, \leq)$ is directed, and for any prime p the diagram

$$\cdots \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/\mathbb{Z} = 1$$

is an inverse system in the category of topological rings, where the maps are the projections and the topologies are discrete. Its limit is the topological ring $\mathbb{Z}_p$ of p**-adic integers**. The topology on $\mathbb{Z}_p$ is *not* discrete. It is an example of a **profinite space**, that is, a topological space arising as an inverse limit of finite discrete spaces. (There are other equivalent definitions.)

The modern replacement for directed ordered sets is **filtered categories**: those in which every finite diagram admits a cone.

Exercises

5.14 What are products and coproducts in an ordered set?

5.15 Let f be an endomorphism of a topological space S. Describe the coequalizer C of $S \underset{1}{\overset{f}{\rightrightarrows}} S$ (in **Top**) as explicitly as possible. When S is the circle S^1, find an f such that C is uncountable but indiscrete.

(When S is the Riemann sphere $\mathbb{C} \cup \{\infty\}$, complex dynamicists call C the set of **grand orbits** under f.)

5.16 Show that monics are stable under pullback: that is, given a pullback square as in Example 5.5, if f_2 is monic then so is p_1.

A map $e : A \longrightarrow B$ is **split epic** if there exists $m : B \longrightarrow A$ such that $e \circ m = 1_B$. Show that split epics are epic.