Lecture 8
Monoidal categories

A monoidal category is a category equipped with some kind of product—not necessarily 'product' in the sense defined earlier. There are two kinds: strict (rarer) and weak (more common).

Definition 8.1 A **strict monoidal category** is a category $\mathcal{V}$ equipped with a functor $\mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ and an object $I \in \mathcal{V}$ satisfying associativity and unit axioms.

The functor is written on objects as

$$(X, Y) \longmapsto X \otimes Y$$

and on maps as

$$\begin{pmatrix} X & & Y \\ \downarrow f & , & \downarrow g \\ X' & & Y' \end{pmatrix} \longmapsto \begin{array}{c} X \otimes Y \\ \downarrow f \otimes g \\ X' \otimes Y'. \end{array}$$

Functoriality of $\otimes$ says that

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g), \qquad 1_X \otimes 1_Y = 1_{X \otimes Y}$$

whenever these make sense. The associativity and unit axioms are that

$$\begin{aligned} (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z), \qquad & X \otimes I = X = I \otimes X, \\ (f \otimes g) \otimes h = f \otimes (g \otimes h), \qquad & f \otimes 1_I = f = 1_I \otimes f \end{aligned}$$

for all objects X, Y, Z and maps f, g, h.

This definition is unnaturally strict as it involves equality of objects. Nevertheless, there are some significant examples.

Example 8.2 Given $n \in \mathbb{N}$, write $\mathbf{n}$ for the n-element set $\{1, \ldots, n\}$ equipped with its usual ordering. Let $\mathbb{D}$ be the category with object-set $\mathbb{N}$ and in which a map $m \longrightarrow n$ is an order-preserving map $\mathbf{m} \longrightarrow \mathbf{n}$; thus, $\mathbb{D}$ is equivalent to the category of finite totally ordered sets. Define $\otimes$ on objects by $m \otimes n = m + n$ and on maps in the evident way; put $I = 0$. Then $(\mathbb{D}, +, 0)$ is a strict monoidal category.

The category usually denoted Δ is the full subcategory of $\mathbb{D}$ obtained by discarding the unit object 0; it is equivalent to the category of finite *nonempty* totally ordered sets.

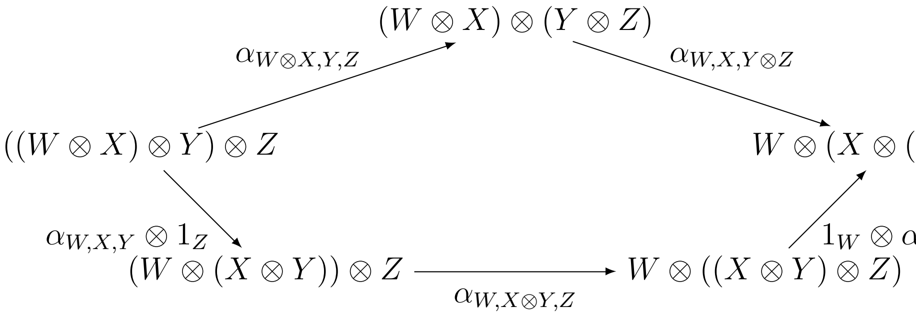
Example 8.3 Any category $\mathcal{A}$ gives rise to a monoidal category $\mathbf{End}(\mathcal{A})$. The underlying category is $[\mathcal{A}, \mathcal{A}]$, the unit object is $1_{\mathcal{A}}$, tensor product on objects is composition of functors, and I leave you to work out what tensor product on maps must be.

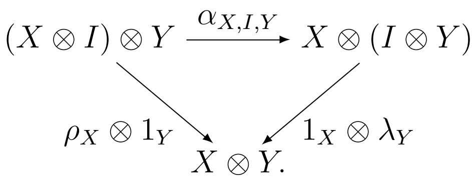
Weak monoidal categories, being the prevalent species, are usually just called 'monoidal categories'.

Definition 8.4 A **monoidal category** is a category $\mathcal{V}$ equipped with a functor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$, an object $I \in \mathcal{V}$, and isomorphisms

$$(X \otimes Y) \otimes Z \xrightarrow[\sim]{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z), \qquad I \otimes X \xrightarrow[\sim]{\lambda_X} X, \qquad X \otimes I \xrightarrow[\sim]{\rho_X} X$$

natural in $X, Y, Z \in \mathcal{V}$ (the **coherence isomorphisms**), such that the following diagrams commute for all $W, X, Y, Z \in \mathcal{V}$:





Loosely, for each $n \in \mathbb{N}$ and each pair of ways of forming the tensor product of n objects we have precisely one isomorphism from the first way to the second. We will come back to this (8.9).

Example 8.5 A strict monoidal category can be regarded as a monoidal category in which all the components of α, λ and ρ are identities. The pentagon and the triangle then commute automatically.

Example 8.6 Let $\mathcal{V}$ be a category in which all finite products exist. Choose a particular terminal object 1, and for each pair (X, Y) of objects, choose a particular product diagram

$$X \xleftarrow{\pi_1^{X,Y}} X \times Y \xrightarrow{\pi_2^{X,Y}} Y.$$

Then there is a canonical way of defining $\times$ on maps so that it becomes a functor $\mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$, and a canonical way of defining natural isomorphisms α, λ, and ρ. This defines a monoidal category.

For instance, let $\mathcal{V} = \mathbf{Set}$, make some sensible definition of ordered pair, and take $X \times Y$ to be the set of ordered pairs with its usual projections onto X and Y; then $\alpha_{X,Y,Z}$ is the map

$$\begin{array}{ccc} (X \times Y) \times Z & \longrightarrow & X \times (Y \times Z), \\ ((x, y), z) & \longmapsto & (x, (y, z)). \end{array}$$

Example 8.7 Dually, any category with finite coproducts gives rise to a monoidal category.

Example 8.8 Similarly, if k is a commutative ring then $(k\text{-}\mathbf{Mod}, \otimes, k)$ is a monoidal category.

Theorem 8.9 (Coherence, Mark 1) *All diagrams commute. More exactly, all diagrams of the same general kind as the pentagon and triangle in 8.4, built up from copies of* α*,* λ *and* ρ*, commute.*

Proof Omitted. $\square$

This tells us that the definition of monoidal category is correct. It also guarantees that, for instance, we can 'identify $X \otimes \mathbb{Z}$ with X' for abelian groups X, safe in the knowledge that there is only one sensible way of identifying the two groups.

The Coherence Theorem concerns the general theory of monoidal categories, not particular monoidal categories. For instance, we may have a monoidal category $\mathcal{V}$ such that $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ for all objects X, Y and Z, but there is no reason why any of the associativity isomorphisms should be the identity: in other words, the diagrams

$$(X \otimes Y) \otimes Z \underset{1}{\overset{\alpha_{X,Y,Z}}{\rightrightarrows}} X \otimes (Y \otimes Z)$$

need not commute.

The notions of functor and natural transformation can be extended to monoidal categories; but in the case of functors, there are several ways to do it.

Definition 8.10 Let $\mathcal{V}$ and $\mathcal{V}'$ be monoidal categories. A **lax monoidal functor** $\mathcal{V} \longrightarrow \mathcal{V}'$ is a functor $F : \mathcal{V} \longrightarrow \mathcal{V}'$ together with a map $\phi_\cdot : I \longrightarrow FI$ and maps

$$\phi_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y)$$

natural in $X, Y \in \mathcal{V}$ (**coherence maps**), satisfying coherence axioms. If $\phi_\cdot$ and $\phi_{X,Y}$ are all isomorphisms then (F, ϕ) is called a **weak monoidal functor**, or **strong monoidal functor**, or just a **monoidal functor**. If $\phi_\cdot$ and $\phi_{X,Y}$ are all identities then (F, ϕ) is a **strict monoidal functor**.

Example 8.11 The forgetful functor U : $\mathbf{Ab} \longrightarrow \mathbf{Set}$ becomes a lax monoidal functor $(\mathbf{Ab}, \otimes, \mathbb{Z}) \longrightarrow (\mathbf{Set}, \times 1)$ via the canonical maps

$$1 \longrightarrow U\mathbb{Z}, \qquad UX \times UY \longrightarrow U(X \otimes Y).$$

There is also a notion of a **monoidal transformation** between monoidal functors. Predictably, monoidal categories $\mathcal{V}$ and $\mathcal{V}'$ are called **monoidally**

equivalent if there exist (weak) monoidal functors $\mathcal{V} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{V}'$ and invertible monoidal transformations $1 \xrightarrow{\sim} G \circ F$, $F \circ G \xrightarrow{\sim} 1$.

Theorem 8.12 (Coherence, Mark 2) *Every monoidal category is monoidally equivalent to some strict monoidal category.*

Proof Omitted. □

This is stronger than Mark 1, as the property 'all diagrams commute' is invariant under monoidal equivalence and holds in any strict monoidal category. It allows us to pretend that every monoidal category is strict, and so write $W \otimes X \otimes Y \otimes Z$ instead of $W \otimes ((X \otimes Y) \otimes Z)$.

* * *

The rest of this lecture concerns two things that you can do with monoidal categories: enrich in them, and take algebraic structures in them.

In homological algebra one deals with **Ab**-categories, that is, categories $\mathcal{A}$ in which each hom-set $\mathcal{A}(A, B)$ carries the structure of an abelian group and composition is bilinear (1.10). In topology one often makes use of the fact that for suitable spaces A and B (say, compactly generated Hausdorff), the set $\mathbf{Top}(A, B)$ carries a natural topology. The general idea is called 'enrichment'; here is the most common formalization.

Definition 8.13 Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a monoidal category. A **category enriched in** $\mathcal{V}$, or $\mathcal{V}$**-category**, $\mathcal{A}$, consists of

- a collection $\mathrm{ob}(\mathcal{A})$
- for each $A, B \in \mathrm{ob}(\mathcal{A})$, an object $\mathcal{A}(A, B) \in \mathcal{V}$
- for each $A, B, C \in \mathrm{ob}(\mathcal{A})$, a map

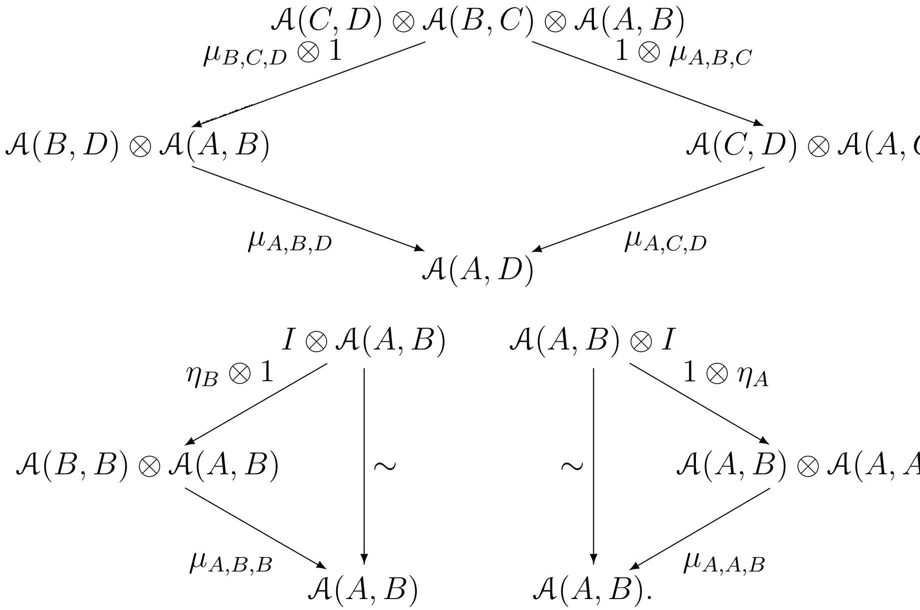
  $$\mu_{A,B,C} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$$

  in $\mathcal{V}$ (**composition**)
- for each $A \in \mathrm{ob}(\mathcal{A})$, a map

  $$\eta_A : I \longrightarrow \mathcal{A}(A, A)$$

  in $\mathcal{V}$ (**identities**)

such that the following diagrams in $\mathcal{V}$ commute for all $A, B, C, D \in \mathrm{ob}(\mathcal{A})$ (the **associativity** and **identity** axioms):



Example 8.14 A category enriched in $(\mathbf{Set}, \times, 1)$ is just an ordinary (locally small) category.

Example 8.15 A category enriched in $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is what is usually called an **Ab**-category; for instance,

k-**Mod** is an **Ab**-category for any commutative ring k.

Almost all the concepts and results of ordinary category theory can be extended to the $\mathcal{V}$-enriched context, assuming that $\mathcal{V}$ is symmetric monoidal closed and has all limits and colimits. **Symmetric** means that $\mathcal{V}$ comes equipped with an isomorphism $\gamma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ for each $X, Y \in \mathcal{V}$, satisfying naturality and coherence axioms. **Closed** means that for each $Y \in \mathcal{V}$, the functor $- \otimes Y : \mathcal{V} \longrightarrow \mathcal{V}$ has a left adjoint, written $[Y, -]$; then

$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(X, [Y, Z])$$

naturally in $X, Y, Z \in \mathcal{V}$.

Example 8.16 If $\mathcal{V}$ is a cartesian closed category then $(\mathcal{V}, \times, 1)$ is symmetric monoidal closed, defining the symmetry maps in the obvious way.

Example 8.17 $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is symmetric monoidal closed: the symmetric structure is obvious, and $[Y, Z]$ is the abelian group of homomorphisms from Y to Z. Also, **Ab** has all limits and colimits. So almost everything in category theory has an **Ab**-enriched analogue.

'Internal algebraic structures' are algebraic structures whose underlying object may not be a set. For instance, let $\mathcal{V}$ be any monoidal category. A **monoid** in $\mathcal{V}$ consists of an object $X \in \mathcal{V}$ together with maps

$$m : X \otimes X \longrightarrow X, \qquad e : I \longrightarrow X$$

such that certain diagrams expressing the associativity and unit axioms commute. (Compare 8.13 and 7.1.)

Example 8.18 A monoid in $(\mathbf{Set}, \times, 1)$ is a monoid in the ordinary sense.

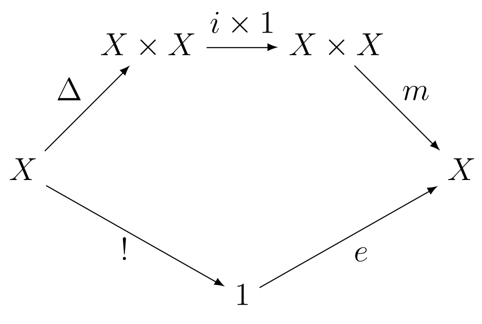
Example 8.19 A monoid in $(\mathbf{Ab}, \otimes, \mathbb{Z})$ consists of an abelian group X together with a bilinear map $m : X \times X \longrightarrow X$ and an element e of X satisfying associativity and unit axioms: in other words, a ring.

Example 8.20 Similarly, a monoid in k-**Mod** is just a k-algebra.

Only a limited range of algebraic structures can be defined inside an arbitrary monoidal category $\mathcal{V}$. But if the monoidal structure comes from ordinary (categorical) product on $\mathcal{V}$ (Example 8.6) then we can define all types of algebraic structure: groups, rings, Lie algebras, k-modules, ... For instance, if $\mathcal{V}$ is a category with finite products then a **group** in $\mathcal{V}$ is an object $X \in \mathcal{V}$ together with maps

$$m : X \times X \longrightarrow X, \qquad i : X \longrightarrow X, \qquad e : 1 \longrightarrow X$$

satisfying commutative diagrams expressing the group axioms. The associativity and unit axioms are as for monoids. To express the inverse axiom '$x^{-1} \cdot x = 1$', write ! for the unique map $X \longrightarrow 1$ and Δ for the diagonal map $(1_X, 1_X) : X \longrightarrow X \times X$; then the axiom is that



commutes.

Example 8.21 A group in **Set** is a group. A group in **Top** is a topological group. A group in the category of smooth manifolds is a Lie group. A group in the category of algebraic varieties is an algebraic group.

Digression 8.22 At this point, the standard remark is that it is impossible to define 'group' in an arbitrary monoidal category, since there is no analogue in an arbitrary monoidal category of the maps $! : X \longrightarrow 1$ and $\Delta : X \longrightarrow X \times X$ used in the definition above. But this is misleading: assuming only that our monoidal category $\mathcal{V}$ is symmetric, it is in fact possible to define 'group in $\mathcal{V}$'.

Thus, a **Hopf algebra** in a symmetric monoidal category $\mathcal{V}$ is an object X equipped with maps

$$m : X \otimes X \longrightarrow X, \qquad e : I \longrightarrow X,$$
$$\delta : X \longrightarrow X \otimes X, \qquad \varepsilon : X \longrightarrow I,$$
$$i : X \longrightarrow X$$

such that certain diagrams commute. A Hopf algebra in k-**Mod** is what is usually called a Hopf algebra over k; an example is the group algebra kG of any group G, where m and e give kG its usual multiplication and unit, and for any $g \in G$ we have

$$\delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad i(g) = g^{-1}.$$

It turns out that a Hopf algebra in **Set** is precisely a group. So Hopf algebras, or 'quantum groups' as they are sometimes called, generalize the concept of group.

Exercises

8.23 Let $\mathcal{V}$ be a symmetric monoidal closed category. There is a contravariant functor $(\ )^* : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{V}$ defined by $X^* = [X, I]$. Exhibit a natural transformation $\alpha : 1_{\mathcal{V}} \longrightarrow (\ )^{**}$; thus, α is to have components $X \longrightarrow X^{**}$. Show by example that α need not be a natural isomorphism.

8.24 Let $\mathcal{V}$ be a category with finite products. Write down the definition of complex vector space in $\mathcal{V}$.