CATEGORY THEORY 3

M2000 — Limits — Tom Leinster

The questions are divided into

- Exercises, to familiarize you with the basics
- Standard questions, which are very roughly of the kind that will make up the exam
- Further questions, which are 'optional' and of varying difficulty and importance.

Some questions require knowledge of other areas of mathematics; skip over any that you haven't the background for. Please report mistakes and obscurities to `leinster@dpmms.cam.ac.uk` .

Exercises

1. Define the terms *limit* and *colimit*. To what extent are limits and colimits unique? Give three examples of each (in specific categories) which did not appear in lectures.

2. Let $D : \mathbb{I} \longrightarrow \mathbf{Set}$ be a diagram over a small category $\mathbb{I}$ in $\mathbf{Set}$. Describe explicitly a limit cone and a colimit cocone for D.

3. Let $\mathcal{C}$ be a category with (chosen) finite products. Show directly that
$$\mathcal{C}(A, B \times C) \cong \mathcal{C}(A, B) \times \mathcal{C}(A, C)$$
naturally in $A, B, C \in \mathcal{C}$.

4. What does it mean for a functor to *preserve*, *reflect* or *create* limits? Give an example of (a) a functor which preserves but does not reflect limits, and (b) a functor which reflects but does not create limits. Show that if $F : \mathcal{C} \longrightarrow \mathcal{D}$ creates limits and $\mathcal{D}$ has limits, then $\mathcal{C}$ also has limits and F preserves them.

5. (i) An object P of a category $\mathcal{C}$ is called *projective* if $\mathcal{C}(P, -) : \mathcal{C} \longrightarrow \mathbf{Set}$ preserves epis. Show that if $\mathbf{Set} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$ is an adjunction in which U preserves epis then $F(S)$ is projective for any set S. Find a non-projective object of $\mathbf{Ab}$.
   (ii) An object I of a category $\mathcal{C}$ is called *injective* if $\mathcal{C}(-, I) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ preserves epis. Show that in $\mathbf{Vect}$ all objects are injective, and find a non-injective object of $\mathbf{Ab}$.
   (iii) An object I of a category $\mathcal{C}$ is called *regular injective* if $\mathcal{C}(-, I)$ preserves regular epis (as defined in q. 7). Let $\mathbf{NVS}$ be the category whose objects are real normed vector spaces and whose morphisms are bounded linear maps of norm ≤ 1. Show that $\mathbb{R}$ is regular injective in $\mathbf{NVS}$.

6. Let k be a field, let $k[X]$ be the vector space of polynomials with coefficients in k, and let $k[\![X]\!]$ be the vector space of formal power series (that is, formal expressions $\sum_{n=0}^{\infty} \lambda_n X^n$) with coefficients in k. Let $(\ )^*$ denote dual vector space. Give a categorical explanation of the fact that $k[X]^* \cong k[\![X]\!]$. (This is an isomorphism of vector spaces, but you need only explain why there is an isomorphism at the level of sets.)

Standard Questions

7. A map $m : A \longrightarrow B$ in a category $\mathcal{C}$ is called *regular monic* if there exist an object C and maps $B \underset{g}{\overset{f}{\rightrightarrows}} C$ such that $A \xrightarrow{m} B \underset{g}{\overset{f}{\rightrightarrows}} C$ is an equalizer diagram. m is called *split monic* if there is a map $e : B \longrightarrow A$ with $e \circ m = 1_A$. *Regular epics* and *split epics* are defined dually.

   (i) Show that split monic $\Rightarrow$ regular monic $\Rightarrow$ monic.

   (ii) In **Ab**, show that all monics are regular, but that not all monics are split.

   (iii) In **Top**, identify the regular monics, and find a monic which is not regular.

   (iv) Prove that in any category, a map is an isomorphism if and only if it is both epic and regular monic. Show by example that a map which is epic and monic need not be an isomorphism.

   (v) Show that epics, regular epics and split epics are all the same in **Set**. (For this you need the axiom of choice, which says *exactly* that epics split in **Set**; of course, this does not hold in every category.)

8. (i) Show that if a category has a terminal object, all binary products and all equalizers, then it has all finite limits.

   (ii) Deduce that if a category has a terminal object and all pullbacks then it has all finite limits. (Hint: consider diagrams

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & B \times B.) \end{tikzcd}$$

   (iii) Let $\mathcal{C}$ be a category with all finite colimits and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor which preserves pushouts and the initial object. Show that F preserves all finite colimits.

   (iv) Show that a functor which preserves pushouts also preserves epics.

9. (i) A *widget* is a set A equipped with a pair of elements $0, 1 \in A$, a ternary operation $[\ ,\ ,\ ] : A^3 \longrightarrow A$, and for each rational number λ a unary operation $\lambda \cdot - : A \longrightarrow A$, satisfying the axioms

$$\begin{aligned} [a, 0, b] &= [\tfrac{3}{4} \cdot b, \tfrac{1}{4} \cdot a, 1] \\ [[a, b, a], c, a] &= 0 \\ \lambda \cdot (\mu \cdot a) &= (\lambda\mu) \cdot a \\ (\lambda + \mu) \cdot a &= [\lambda \cdot a, \mu \cdot a, (5 - \mu) \cdot a] \end{aligned}$$

   for all $a, b, c \in A$ and rational λ, μ. Let **Widget** be the category of widgets and their homomorphisms. Show that the forgetful functor $U : \textbf{Widget} \longrightarrow \textbf{Set}$ creates limits. Deduce that **Widget** has all limits and U preserves them.

   (ii) A *chad* is a set A equipped with an element $0 \in A$ and a ternary operation $[\ ,\ ,\ ] : A^3 \longrightarrow A$, satisfying the axiom

$$[[a, a, a], a, a] = 0$$

   for all $a \in A$. Show that the forgetful functor $\textbf{Widget} \longrightarrow \textbf{Chad}$ preserves and creates limits.

The theory of widgets is a typical finitary algebraic theory, and your proof of (i) should apply equally to categories of algebras such as rings, groups, Lie algebras, modules, etc etc. Similarly, your proof of (ii) should apply to all functors between such categories which arise from forgetting some of a theory.

10. Let $\mathbb{I}$ be a small category. Show that a category $\mathcal{C}$ has all colimits of shape $\mathbb{I}$ if and only if the diagonal functor $\Delta : \mathcal{C} \longrightarrow [\mathbb{I}, \mathcal{C}]$ has a left adjoint.

11. Let $\mathbb{C}$ be a small category.
    (i) What does it mean to say that limits and colimits are computed pointwise in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$? Prove that this is so.
    (ii) Characterize the monics and epics in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. (Now see if you can do this without the aid of (i).)

12. Let $\mathbb{C}$ be a small category.
    (i) Show that for each $A \in \mathbb{C}$, the representable functor $H^A : \mathbb{C} \longrightarrow \mathbf{Set}$ preserves limits.
    (ii) Show that the Yoneda embedding $H_\bullet : \mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ also preserves limits.

13. (i) Let $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ be an adjunction. Show that F preserves all colimits which exist in $\mathcal{C}$, and G all limits which exist in $\mathcal{D}$.
    (ii) Show that the forgetful functor from groups to sets has no right adjoint.
    (iii) Deduce from (i) that the chain of adjoints $C \dashv D \dashv O \dashv I$ in sheet 2, question 5 extends no further.

14. Let $\mathbb{C}$ be a small category.
    (i) Prove that any presheaf $X \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is a colimit of representable presheaves.
    (ii) What does it mean for a category to be *cartesian closed*? Show that $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed. (You may assume that products exist and are computed pointwise in a presheaf category.)
    (iii) Now suppose that $\mathbb{C}$ is cartesian closed. Prove that the Yoneda embedding $H_\bullet$ preserves the cartesian closed structure—that is, $H_\bullet$ preserves finite products and exponentials.

Further Questions

15. Let $\mathcal{C}$ be a category. The statement 'monics in $\mathcal{C}$ are closed under composition' means that if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are both monic, then $g \circ f : A \longrightarrow C$ is also monic. The statement 'monics in $\mathcal{C}$ are stable under pullback' means that whenever

$$\begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

is a pullback square in $\mathcal{C}$, then f monic $\Rightarrow$ f' monic. Similar terminology can be used for any other class of morphisms in $\mathcal{C}$.

Determine whether each of the following classes of morphisms—

monics,	regular monics,	split monics,
epics,	regular epics,	split epics

—is closed under composition, and whether it is stable under pullback. So there are 12 parts to this question, and your answer to each should be one of the following:

- yes, in all categories
- in **Set**, but not in all categories
- not even in **Set**.

Some hints:

- let $\mathcal{C}$ be the full subcategory of **Ab** whose objects are those abelian groups with no element of order 4, and consider the map $2 \times - : \mathbb{Z} \longrightarrow \mathbb{Z}$ (which can be composed with itself)
- let $\mathbf{2} = (\bullet \longrightarrow \bullet)$ be the category with two objects and a single map between them, let $\mathbf{I} = (\bullet \rightleftarrows \bullet)$ be the category with two objects and a single isomorphism between them, and consider the two non-constant maps from $\mathbf{2}$ to $\mathbf{I}$.

16. *Extremely optional, way off the syllabus, and probably quite time-consuming: more a project than an exercise. However, you might find it useful in relation to other courses. Don't expect your supervisor to mark it!*

This is a general construction of (co)homology theories, including group homology and cohomology, Ext and Tor, and sheaf cohomology. Your task is to understand and flesh out the skeletal description below. Help can be found in Chapter 2 of *An Introduction to Homological Algebra*, Charles A. Weibel, CUP 1994.

An *additive category* is a category $\mathcal{A}$ equipped with the structure of an abelian group on each hom-set $\mathcal{A}(A, B)$, in such a way that composition is bilinear (that is, $g \circ (f + f') = g \circ f + g \circ f'$ and dually). An *abelian category* is an additive category $\mathcal{A}$ satisfying certain axioms, which say that we can speak of kernels and cokernels of maps in $\mathcal{A}$ and that they behave more or less as in the category of modules over a commutative ring. This means that we can also manipulate exact sequences in $\mathcal{A}$ in the familiar way.

A *projective resolution* of an object A of an abelian category $\mathcal{A}$ is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \qquad (*)$$

in which P_n is a projective object for each $n \in \mathbb{N}$. We say that $\mathcal{A}$ has *enough projectives* if every object has a projective resolution. Dually *injective resolutions*, which go the other way:

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots.$$

An *additive functor* between additive categories is a functor preserving addition of morphisms. An additive functor is *right exact* if it preserves finite colimits, and dually *left exact*.

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories, and suppose that $\mathcal{A}$ has enough projectives. For each $A \in \mathcal{A}$, choose a projective resolution $(*)$ for A. This gives rise to a complex

$$\cdots \longrightarrow F(P_2) \longrightarrow F(P_1) \longrightarrow F(P_0) \longrightarrow 0,$$

and we define $(L_nF)(A)$ to be the nth homology of this complex: that is,

$$(L_nF)(A) = \ker(F(P_n) \longrightarrow F(P_{n-1}))/\mathrm{im}(F(P_{n+1}) \longrightarrow F(P_n)).$$

Since F is right exact, $(L_0F)(A) \cong F(A)$.

It turns out that $(L_nF)(A)$ is independent (up to isomorphism) of the choice of projective resolution, and that L_nF can be defined on morphisms in a natural way. The functors

$$L_0F \cong F,\ L_1F,\ L_2F,\ \ldots : \mathcal{A} \longrightarrow \mathcal{B}$$

are called the *left derived functors* of F. (They have a certain universal property: Weibel, Theorem 2.4.7.) Dually, a left exact functor has right derived functors R^nF, as long as there are enough injectives. Here are some examples.

(i) Fix a group G. An object of $[G, \mathbf{Ab}]$ is called a *G-module.*

- The diagonal functor $\mathbf{Ab} \longrightarrow [G, \mathbf{Ab}]$ has a left adjoint $(-)_G$, where

  $$A_G = A/\text{subgroup generated by } \{ga - a \mid g \in G, a \in A\}$$

  (compare sheet 2, question 12). Being a left adjoint, $(-)_G$ is right exact, and therefore has left derived functors $H_n(G, -)$. This is *group homology*; of particular interest is $H_n(G, \mathbb{Z})$, where $\mathbb{Z}$ is acted on trivially by G.
- The diagonal functor also has a right adjoint $(-)^G$, where

  $$A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}.$$

  The right derived functors give *group cohomology*, $H^n(G, -)$.

(ii) Fix a commutative ring R and an R-module A. We have an adjunction

$$(A \otimes -) \dashv \mathrm{Hom}_R(A, -) : R\text{-}\mathbf{Mod} \longrightarrow R\text{-}\mathbf{Mod}$$

between functors on the category of R-modules. Then:

- $A \otimes -$ is a left adjoint, so has left derived functors $\mathrm{Tor}_n(A, -)$
- $\mathrm{Hom}_R(A, -)$ is a right adjoint, so has right derived functors $\mathrm{Ext}^n(A, -)$.

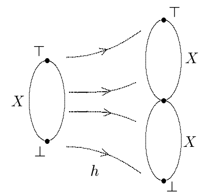
(This is one of various contexts in which Ext and Tor can be defined.)

(iii) Fix a space X, and let $\mathbf{Sh}(X)$ be the category of sheaves of abelian groups on X. This is a reflective subcategory of the category $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Ab}]$ of presheaves of abelian groups on X, so it follows from the argument of sheet 2, question 5 that the global sections functor $\Gamma : \mathbf{Sh}(X) \longrightarrow \mathbf{Ab}$ has a left adjoint (the 'constant sheaf' functor). The right derived functors of Γ are the *sheaf cohomology* functors $H^n(X, -)$. The constant sheaf $\mathbb{Z}$ is of particular interest: for many spaces X, $H^n(X, \mathbb{Z})$ coincides with the ordinary cohomology of X.

17. *Again highly optional; more arithmetic and elementary real analysis than category theory. We provide a universal characterization of the real numbers. More exactly, starting from a very primitive notion of 'bisection', we construct a category in which the closed real interval is the terminal object.*

A *bipointed set* is a set X equipped with an element $\perp$ ('bottom') and an element $\top$ ('top'), such that $\perp \neq \top$. A *map of bipointed sets* is a function preserving top and bottom elements. This defines a category $\mathbf{Bip}$.

A *self-bisecting set* is a bipointed set $(X, \perp, \top)$ equipped with functions $h_-, h_+ : X \longrightarrow X$ such that



- $h_-(\bot) = \bot$
- for all $x \in X$, $h_+(x) = \bot$ or $h_-(x) = \top$
- $h_+(\top) = \top$.

A *map* $f : (X, h) \longrightarrow (Y, k)$ *of self-bisecting sets* is a map $f : X \longrightarrow Y$ in **Bip** such that $f \circ h_- = k_- \circ f$ and $f \circ h_+ = k_+ \circ f$. This defines a category **SBS**.

The real interval $I = [-1, +1]$, with bottom -1 and top $+1$, is a bipointed set. Define $j_-, j_+ : I \longrightarrow I$ by

$$j_-(t) = \begin{cases} 2t+1 & \text{for } t \in [-1, 0] \\ 1 & \text{for } t \in [0, +1] \end{cases}, \qquad j_+(t) = \begin{cases} -1 & \text{for } t \in [-1, 0] \\ 2t-1 & \text{for } t \in [0, +1] \end{cases}.$$

Then (I, j) is a self-bisecting set (check), and we will prove that it is the *terminal* one.

Let $(X, h) \in$ **SBS**. Call an element x of X

- *lower* if $h_-(x) \neq \top$ (in which case $h_+(x) = \bot$)
- *middle* if $h_-(x) = \top$ and $h_+(x) = \bot$
- *upper* if $h_+(x) \neq \bot$ (in which case $h_-(x) = \top$);

precisely one of these holds for each x.

(i) Show that a map $f : X \longrightarrow I$ in **Bip** defines a map $(X, h) \longrightarrow (I, j)$ in **SBS** if and only if
  - for x lower, $f(h_-(x)) = 2f(x) + 1$
  - for x middle, $f(x) = 0$
  - for x upper, $f(h_+(x)) = 2f(x) - 1$.

For each $x \in X$, define a sequence (x_n) in X and sequences $(l_n(x)), (u_n(x))$ in I by

$$x_0 = x, \qquad l_0(x) = -1, \qquad u_0(x) = +1,$$

and

$$\begin{array}{llll} \text{for } x_n \text{ lower}, & x_{n+1} = h_-(x_n), & l_{n+1}(x) = l_n(x), & u_{n+1}(x) = \frac{1}{2}(l_n(x) + u_n(x)); \\ \text{for } x_n \text{ middle}, & x_{n+1} = x_n, & l_{n+1}(x) = \frac{1}{2}(l_n(x) + u_n(x)), & u_{n+1}(x) = \frac{1}{2}(l_n(x) + u_n(x)); \\ \text{for } x_n \text{ upper}, & x_{n+1} = h_+(x_n), & l_{n+1}(x) = \frac{1}{2}(l_n(x) + u_n(x)), & u_{n+1}(x) = u_n(x). \end{array}$$

(ii) Let $x \in X$ and $y = x_1$. Show that $y_n = x_{n+1}$ for all $n \geq 0$, and that
  - for x lower, $l_n(y) = 2l_{n+1}(x) + 1$ and $u_n(y) = 2u_{n+1}(x) + 1$
  - for x middle, $l_{n+1}(x) = 0 = u_{n+1}(x)$
  - for x upper, $l_n(y) = 2l_{n+1}(x) - 1$ and $u_n(y) = 2u_{n+1}(x) - 1$

  for all $n \geq 0$.

(iii) Let $x \in X$. Show that $(l_n(x))$ is an increasing sequence, $(u_n(x))$ is decreasing, $l_n(x) \leq u_n(x)$ for all n, and that the two sequences converge to a common value $g(x)$ in I. Deduce from (i) and (ii) that g is a map $(X, h) \longrightarrow (I, j)$ in **SBS**.

(iv) Also deduce from (i) and (ii) that if $f : (X, h) \longrightarrow (I, j)$ is a map in **SBS** then $l_n(x) \leq f(x) \leq u_n(x)$ for all n and x, and so $f = g$.

Conclude that the real interval I is the terminal self-bisecting set.