The categorical origins of Lebesgue integration

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These slides: on my web page

Broad context

- One function of category theory: to show that in many cases,

  that which is socially important is categorically natural.

- Recent work of Tel Aviv analysts (Milman, Artstein-Avidan, Alesker, . . .), showing that various famous constructions are uniquely characterized by their elementary properties. E.g.:
  - the Fourier transform is unique such that . . .
  - the Legendre transform is unique such that . . .

  (Related to this talk only spiritually.)

Plan

Theorem A Universal characterization of the space

$$L^1[0,1] = \frac{\{\text{Lebesgue-integrable functions } [0,1] \to \mathbb{R}\}}{\text{equality almost everywhere}}$$

and resulting unique characterization of $\int_0^1$.

Theorem B Universal characterization of the functor

$$L^1 \colon (\text{measure spaces}) \to (\text{Banach spaces})$$

and resulting unique characterization of $\int$ on arbitrary measure spaces.

Theorem A: $[0, 1]$

The undergraduate-textbook approach

To define the space of Lebesgue-*integrable* functions on $[0, 1]$:

- define null set (set of measure zero), hence almost everywhere
- define step function (finite linear combination of characteristic functions of intervals)
- define $\mathcal{L}^{\text{inc}}[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is an almost everywhere limit of some increasing sequence of step functions}\}$
- define $\mathcal{L}^1[0,1] = \{f - g \mid f, g \in \mathcal{L}^{\text{inc}}[0,1]\}$
- define $L^1[0,1] = \mathcal{L}^1[0,1]/(\text{equality almost everywhere})$.

To define Lebesgue *integration*:

- for a step function $f = \sum_i c_i I_{A_i}$ (where the A_i are intervals, $c_i \in \mathbb{R}$, and I means characteristic function), put $\int f = \sum_i c_i \cdot \text{length}(A_i)$
- extend to $L^1[0,1]$ by continuity and linearity.

Alternative approach: $L^1[0,1]$ is the Banach space completion of $\{\text{continuous functions } [0,1] \to \mathbb{R}\}$ with norm $\|f\| = \int |f|$. But for that, we need to already know how to integrate continuous functions.

We will leap over all these preliminary definitions.

Conceptual background to the theorem

The topological space $[0,1]$ has *exactly* the structure needed to do elementary homotopy theory, if we define a 'path' in a space X to be a continuous map $[0,1] \to X$.

That is, the topological space $[0,1]$ comes equipped with:

- two distinct, closed, basepoints, 0 and 1 (so each path has a beginning and an end, possibly distinct)
- a map $[0,1] \xrightarrow{\times 2} \frac{[0,1] \sqcup [0,1]}{\text{first } 1 \sim \text{second } 0}$ (so paths can be concatenated)

and:

Theorem (Freyd; Leinster) $[0,1]$ *is terminal as such.*

We'll prove a kind of dual: $L^1[0,1]$, with a little extra structure, is initial.

Set-up

Recall: a Banach space is a complete normed vector space.

Conventions:

- For concreteness, we'll work over $\mathbb{R}$ (though everything works equally over $\mathbb{C}$).
- A map $\phi\colon X \to Y$ of Banach spaces is a linear map such that $\|\phi(x)\| \leq \|x\|$ for all $x \in X$.
- $X \oplus Y$ has norm $\|(x, y)\| = \frac{1}{2}(\|x\| + \|y\|)$.

Let $\mathscr{A}$ be the category of triples (X, u, ξ) where:

- X is a Banach space
- $u \in X$ with $\|u\| \leq 1$ (or equivalently, $u\colon \mathbb{R} \to X$)
- $\xi\colon X \oplus X \to X$ with $\xi(u, u) = u$

and with the obvious maps.

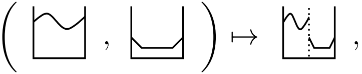
E.g.: $(\mathbb{R}, 1, \mathrm{mean}) \in \mathscr{A}$.

Universal characterization of Lebesgue integrability

Theorem A *The initial object of* $\mathscr{A}$ *is* $\big(L^1[0,1], I_{[0,1]}, \gamma\big)$*, where* $I_{[0,1]}$ *is the constant function* 1 *and*

$$\gamma\colon L^1[0,1] \oplus L^1[0,1] \to L^1[0,1]$$

is



i.e.

$$(\gamma(f,g))(t) = \begin{cases} f(2t) & \text{if } t < 1/2 \\ g(2t-1) & \text{if } t > 1/2 \end{cases}$$

($f, g \in L^1[0,1]$*,* $t \in [0,1]$*).*

This characterization of $L^1[0,1]$ needs none of the classical preliminary definitions—only the concepts of Banach space and mean.

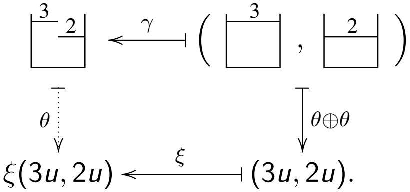
Universal characterization of Lebesgue integrability

Theorem A *The initial object of* $\mathscr{A}$ *is* $\left(L^1[0,1], I_{[0,1]}, \gamma\right)$ *where* $I_{[0,1]}$ *is the constant function* 1 *and* γ *is 'juxtapose and squeeze'.*

Proof (sketch) Given $(X, u, \xi) \in \mathscr{A}$, we must construct unique θ such that

$$\begin{array}{ccccc}
\mathbb{R} & \xrightarrow{I_{[0,1]}} & L^1[0,1] & \xleftarrow{\gamma} & L^1[0,1] \oplus L^1[0,1] \\
\| & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta\oplus\theta} \\
\mathbb{R} & \xrightarrow{u} & X & \xleftarrow{\xi} & X \oplus X
\end{array}$$

commutes. LH square tells us θ on constants. RH square then gives (e.g.)



Repeating argument tells us θ on step functions with dyadic breakpoints. Then use continuity.

Universal characterization of Lebesgue integrability

Theorem A *The initial object of* $\mathscr{A}$ *is* $\left(L^1[0,1], I_{[0,1]}, \gamma\right)$ *where* $I_{[0,1]}$ *is the constant function* 1 *and* γ *is 'juxtapose and squeeze'.*

Alternative proof (sketch) Let $\mathbf{Ban}_* = \mathbb{R}/\mathbf{Ban}$, the category of pointed Banach spaces.

There is an endofunctor T of $\mathbf{Ban}_*$ given by

$$T(X, u) = \big(X \oplus X, (u, u)\big).$$

Then $\mathscr{A} = T$-$\mathbf{Alg}$.

We can *construct* the initial T-algebra using Adámek's theorem: it's the colimit of

$$\mathbb{R} \to T(\mathbb{R}) \to T^2(\mathbb{R}) \to T^3(\mathbb{R}) \to \cdots ,$$

which is $L^1[0,1]$.

Unique characterization of Lebesgue integr*ation*

By initiality, there is a unique map

$$\left(L^1[0,1], I_{[0,1]}, \gamma\right) \to (\mathbb{R}, 1, \text{mean})$$

in $\mathbb{R}$. This is $\int_0^1$.

In other words:

Corollary (old) *$\int_0^1$ is the unique bounded linear map $L^1[0,1] \to \mathbb{R}$ satisfying:*

- $\displaystyle\int_0^1 I_{[0,1]} = 1$
- $\displaystyle\int_0^1 f = \frac{1}{2}\left\{\int_0^1 f\left(\tfrac{1}{2}t\right)dt + \int_0^1 f\left(\tfrac{1}{2}(t+1)\right)dt\right\}.$

Summary

$L^1[0,1]$ is the universal Banach space equipped with a small amount of elementary extra structure.

Theorem B: Arbitrary measure spaces

Background: measure spaces

Recall: A (finite) measure space consists of:

- a set M
- a collection of subsets of M, called measurable sets
- a function $\mu_M \colon \{\text{measurable subsets of } M\} \to [0, \infty)$,

satisfying axioms.

Let $M = (M, \mu_M)$ and $N = (N, \mu_N)$ be measure spaces.

An embedding of M into N is an injection $j \colon M \to N$ such that whenever $A \subseteq M$ is measurable,

- $jA \subseteq N$ is measurable, and
- $\mu_N(jA) = \mu_M(A)$.

Write $\mathbf{Meas}_{\mathbf{emb}}$ for the category of measure spaces and embeddings.

Background: integrable functions on measure spaces

For each measure space M, there is a Banach space

$$L^1(M) = (\text{integrable functions } M \to \mathbb{R})/(\text{equality almost everywhere}).$$

More exactly:

- we have a functor $L^1 \colon \mathbf{Meas_{emb}} \to \mathbf{Ban}$ (where given $j \colon M \to N$, the resulting map $j_* \colon L^1(M) \to L^1(N)$ is extension by zero)
- we have for each M an element $I_M \in L^1(M)$ (the constant function 1).

Some properties:

- $\|I_M\| \leq \mu_M(M)$ for all M (indeed, they're equal!)

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- we have for each M an element $I_M \in L^1(M)$ (the constant function 1).

Some properties:

- $\|I_M\| \leq \mu_M(M)$ for all M
- if $N \xrightarrow{j} M \xleftarrow{j'} N'$ with $jN \sqcup j'N' = M$, then $j_* I_N + j'_* I_{N'} = I_M$.

Theorem (informally): *L^1 is universal as such.*

Universal characterization of integrability

Let $\mathscr{B}$ be the category of pairs (F, u) where:

- F is a functor $\mathbf{Meas_{emb}} \to \mathbf{Ban}$
- u assigns to each measure space M an element $u_M \in F(M)$

such that

- $\|u_M\| \leq \mu_M(M)$ for all M
- if $N \xrightarrow{j} M \xleftarrow{j'} N'$ with $jN \sqcup j'N' = M$, then $(Fj)u_N + (Fj')u_{N'} = u_M$

and with the obvious maps.

Theorem B: *The initial object of $\mathscr{B}$ is (L^1, I).*

Again, this entirely bypasses the usual preliminary definitions.

Universal characterization of integrability

The proof goes via an intermediate step.

Define $\mathscr{B}'$ just as we defined $\mathscr{B}$, but with normed vector spaces in place of Banach spaces.

Thus, $\mathscr{B}'$ is the category of pairs (F, u) where:

- F is a functor $\mathbf{Meas_{emb}} \to$ (normed vector spaces)
- u assigns to each measure space M an element $u_M \in F(M)$

such that

- $\|u_M\| \leq \mu_M(M)$ for all M
- if $N \xrightarrow{j} M \xleftarrow{j'} N'$ with $jN \sqcup j'N' = M$, then $(Fj)u_N + (Fj')u_{N'} = u_M$

and with the obvious maps.

Universal characterization of integrability

The proof goes via an intermediate step.

Define $\mathscr{B}'$ just as we defined $\mathscr{B}$, but with normed vector spaces in place of Banach spaces.

Proposition *The initial object of* $\mathscr{B}'$ *is* (Simp, I), *where* $\mathrm{Simp}(M)$ *is the set of simple functions on* M *(finite linear combinations of characteristic functions of measurable subsets)*.

Sketch proof: Given $(F, u) \in \mathscr{B}'$, we must construct for each M a map $\theta_M \colon \mathrm{Simp}(M) \to F(M)$. It has to be given by

$$\theta_M\left(\sum_i c_i I_{A_i}\right) = \sum_i c_i u_{A_i}$$

($A_i \subseteq M$ measurable, $c_i \in \mathbb{R}$). Check that this 'definition' of θ_M is consistent, etc. □

The theorem then follows by continuity.

Unique characterization of integr*ation*

$\mathscr{B}$ has an object $(\mathbb{R}, \text{tot})$, where $\mathbb{R}\colon \textbf{Meas}_{\textbf{emb}} \to \textbf{Ban}$ has constant value $\mathbb{R}$ and $\text{tot}_M = \mu_M(M) \in \mathbb{R}$ for each M.

By initiality, there is a unique map

$$(L^1, I) \to (\mathbb{R}, \text{tot})$$

in $\mathscr{B}$. This is the natural transformation $\left(L^1(M) \xrightarrow{\int_M} \mathbb{R}\right)_{\text{measure spaces } M}$.

In other words:

Corollary *The family of bounded linear maps* $\left(L^1(M) \xrightarrow{\int_M} \mathbb{R}\right)_M$ *is uniquely characterized by:*

- $\int_M 1 = \mu_M(M)$ *for all* M
- *whenever* $f \in L^1(M)$ *and we have an embedding* $M \to N$, *then* $\int_N (f \text{ extended by zero}) = \int_M f$.

Variants

Can (attempt to) vary the theorem along several axes:

- allow infinite measure spaces
- allow signed measures
- incorporate measure-preserving maps between measure spaces (with respect to which L^1 is *contra*variant).

Regarding the last: there is a similar universal characterization of the functor

$$L^1 \colon \mathbf{Meas}^{\mathrm{op}} \to \mathbf{Ban}$$

where the maps $N \to M$ in **Meas** are the measure-preserving partial maps (e.g. measure-preserving total maps $N \to M$ or embeddings $M \to N$).

Conclusion

Summary

$L^1[0,1]$ is the initial Banach space equipped with a small amount of elementary extra structure.

The universal property of $L^1[0,1]$ gives a unique characterization of $\int_0^1$.

L^1 is the initial functor (measure spaces) $\rightarrow$ (Banach spaces) equipped with a small amount of elementary extra structure.

The universal property of L^1 gives a unique characterization of $\int$ on measure spaces.

Question

Roughly, Theorem B states that L^1 is universal among functors

$$\text{(measure spaces)} \to \text{(Banach spaces)}.$$

But why should we want to turn a measure space into a Banach space?

In other words: Theorem B states precisely the idea that *once we have decided* to turn measure spaces into Banach spaces, L^1 is the natural way to do it.

But can we state precisely the idea that turning a measure space into a Banach space is a natural thing to do?