The Cardinality of a Metric Space

Tom Leinster (Glasgow/EPSRC)

Parts joint with Simon Willerton (Sheffield)

Where does the idea come from?

enriched categories

categories

metric spaces

Where does the idea come from?

enriched categories

↪ ↩

finite categories

every finite category A
has a cardinality
(or Euler characteristic)
$|A|$

metric spaces

Where does the idea come from?

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But where does *this* idea come from?
See two papers
listed on my web page

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1. The cardinality of a finite metric space

Definition

Let $A = \{a_1, \ldots, a_n\}$ be a finite metric space.
Write Z for the $n \times n$ matrix with $Z_{ij} = e^{-2d(a_i, a_j)}$.

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In principle, Z is defined by $Z_{ij} = C^{d(a_i,a_j)}$ for some constant C.
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Warning (Tao)

There exist finite metric spaces whose cardinality is undefined (i.e. with Z non-invertible).

Reference

'Metric spaces', post at *The n-Category Café*, 9 February 2008

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Example (two-point spaces)

Let $A = (\bullet \overset{\leftarrow\ d\ \rightarrow}{} \bullet)$. Then

$$Z = \begin{pmatrix} e^{-2\cdot 0} & e^{-2\cdot d} \\ e^{-2\cdot d} & e^{-2\cdot 0} \end{pmatrix} = \begin{pmatrix} 1 & e^{-2d} \\ e^{-2d} & 1 \end{pmatrix},$$

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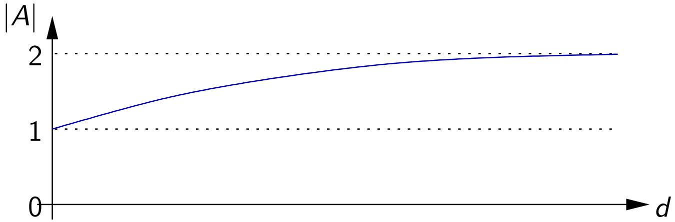

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$$\chi_A : (0, \infty) \longrightarrow \mathbb{R}, \qquad t \longmapsto |tA|.$$

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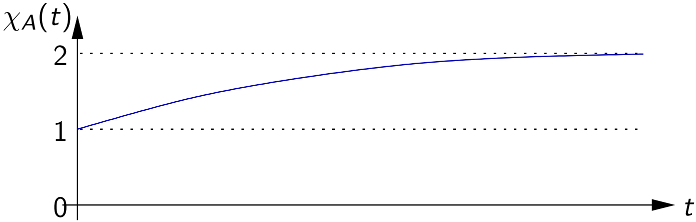
$A = (\bullet \xleftarrow{} 1 \xrightarrow{} \bullet)$:



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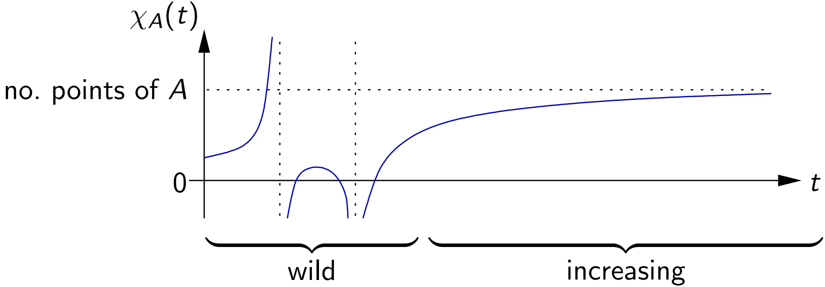
Generic example



2. Some geometric measure theory

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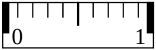
Suppose we want a ruler of length $1\,\mathrm{cm}$: 

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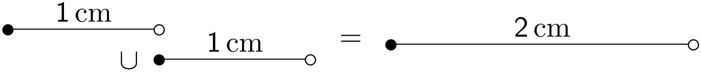
A half-open interval is good:



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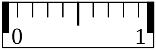
Suppose we want a ruler of length $1\,\mathrm{cm}$: 

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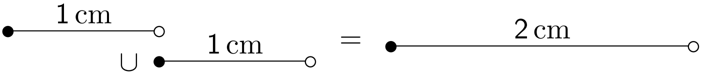

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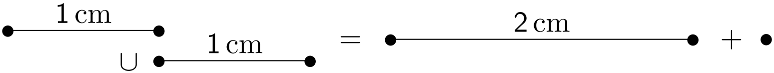

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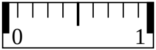
Suppose we want a ruler of length $1\,\mathrm{cm}$: 

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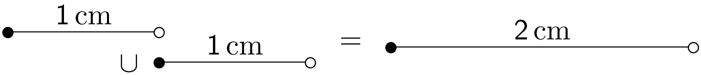

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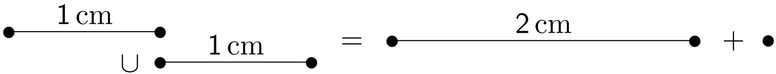

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$$\mathsf{size}([0,1]) = 1\,\mathrm{cm} + 1\,\mathsf{point} = 1\,\mathrm{cm}^1 + 1\,\mathrm{cm}^0 = 1\,\mathrm{cm} + 1.$$

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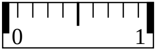
Suppose we want a ruler of length $1\,\mathrm{cm}$: 

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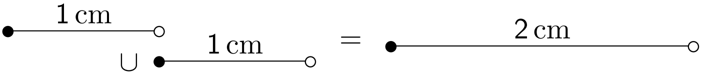

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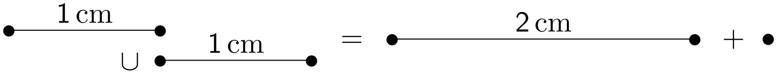

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In general,

$$\boxed{\mathsf{size}([0,\ell]) = \ell\,\mathrm{cm} + 1.}$$

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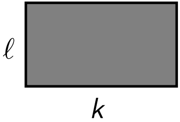
Examples

- Size of rectangle 

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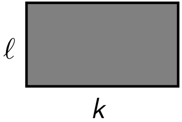
- Size of rectangle 

is

area

$\frac{1}{2}\times$perimeter

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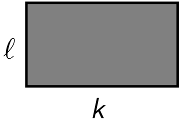
- Size of rectangle 

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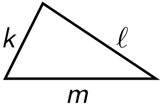
- Size of hollow triangle 

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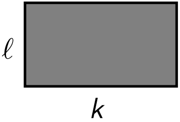
- Size of rectangle  is

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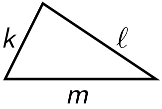
- Size of hollow triangle  is

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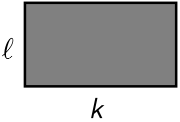
- Size of rectangle 

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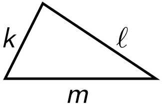
- Size of hollow triangle 

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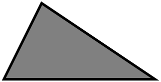
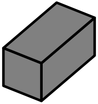
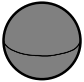
- Similarly, can compute sizes of  ,  ,  , ...

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Hadwiger's Theorem says that there are essentially $n+1$ such measures. They are called the intrinsic volumes,

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$$\begin{aligned} \mu_0 &= \text{Euler characteristic} \\ \mu_1 &= \text{perimeter} \\ \mu_2 &= \text{area.} \end{aligned}$$

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Remark

This is the reason for the choice of the constant e^{-2}.

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Remark

$[0, \ell]$ has cardinality function $t \mapsto |\,[0, t\ell]\,| = \ell t + 1$: so '$t = \mathrm{cm}$'.

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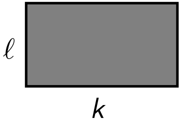
Example (rectangle)

With this metric, $[0,k] \times [0,\ell] =$ 

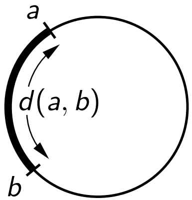
has cardinality function

$$t \mapsto (kt+1)(\ell t+1) = k\ell\, t^2 + (k+\ell)t + 1.$$

3. The cardinality of a compact metric space

Example (circle)

Let C_ℓ be the circle of circumference ℓ, with metric given by length of shortest arc.



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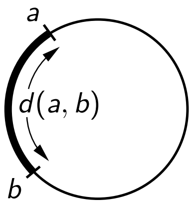

Then

$$|C_\ell| = \frac{\ell}{1 - e^{-\ell}} = \sum_{n=0}^{\infty} B_n \frac{(-\ell)^n}{n!}$$

where B_n is the nth Bernoulli number.

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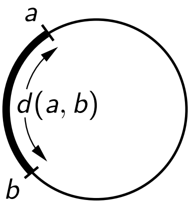

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Asymptotics

- $|C_\ell| \to 1$ as $\ell \to 0$.

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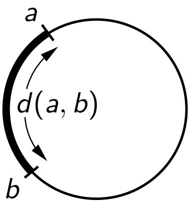
Let C_ℓ be the circle of circumference ℓ, with metric given by length of shortest arc.



Then

$$|C_\ell| = \frac{\ell}{1 - e^{-\ell}} = \sum_{n=0}^{\infty} B_n \frac{(-\ell)^n}{n!}$$

where B_n is the nth Bernoulli number.

Asymptotics

- $|C_\ell| \to 1$ as $\ell \to 0$.
- $|C_\ell| - \ell \to 0$ as $\ell \to \infty$

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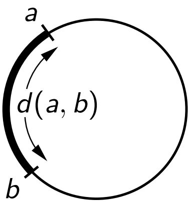

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Asymptotics

perimeter

Euler char

- $|C_\ell| \to 1$ as $\ell \to 0$.
- $|C_\ell| - \ell \to 0$ as $\ell \to \infty$: so when ℓ is large, $|C_\ell| \approx \ell + 0$.

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Hypothesis

All the important invariants of compact metric spaces can be derived from cardinality

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Examples

- Euler characteristic

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- Euler characteristic
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- Hausdorff dimension

Review

metric spaces
as enriched categories

cardinality (Euler char)
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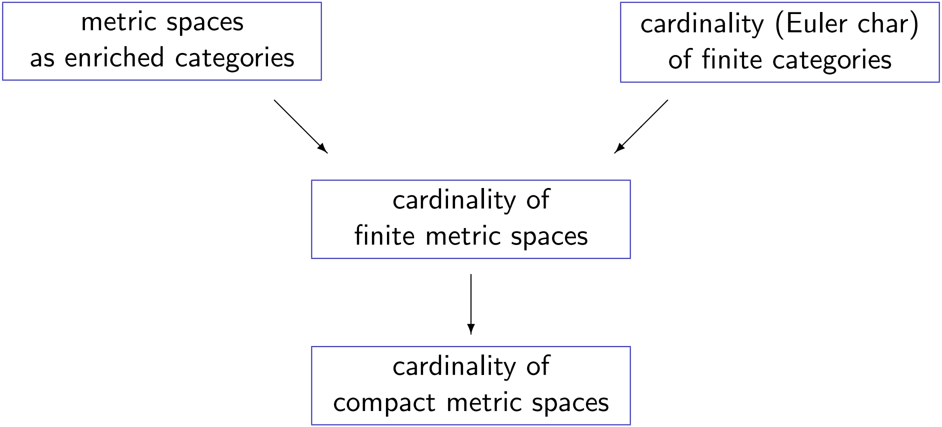
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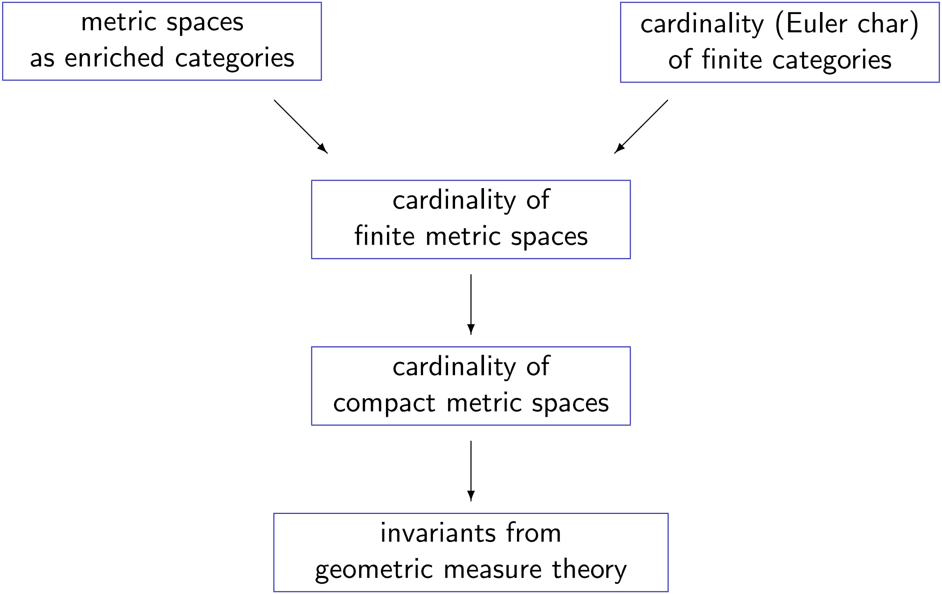
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invariants from geometric measure theory

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Example of a diversity measure

'Effective number of species'

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'Effective number of species' = cardinality of the metric space of species

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