The place of diversity in pure mathematics

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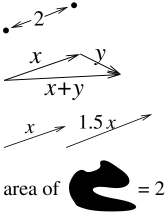
Mathematics in the early 20th century: a ‘just so’ story

Pseudo-history

There are many things you can do in the plane...

measure distance between points:		⇝	metric spaces
add vectors:		⇝	groups
scale:		⇝	vector spaces
measure area:		⇝	measure spaces

Realization It's useful to study each aspect in isolation.

Example Consider all length-30 sequences of symbols A, G, C or T, such as

CGGATACCGTACTAATCCCAGGTTACAACT.

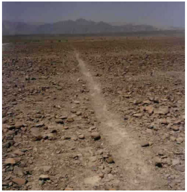
We could define the distance between two sequences to be the number of places where they differ. This gives an example of a 'metric space'.

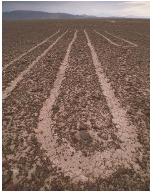
(But we can't add sequences together, or scale them, or measure the area of a set of sequences.)

Pseudo-history

Consequence of this approach: Mathematics split into many subdisciplines.

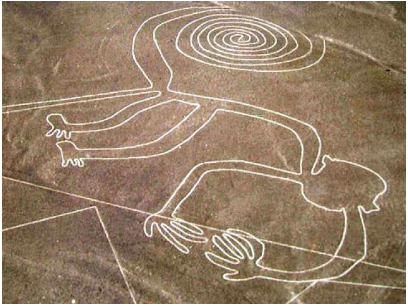
Danger: With many separate lines of development. . .





. . . we fail to see how they link up.

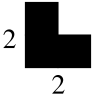
Category theory (my subject) provides a bird's eye view:

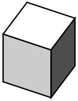


How big is a thing?

Notions of size

There are many notions of size in mathematics:

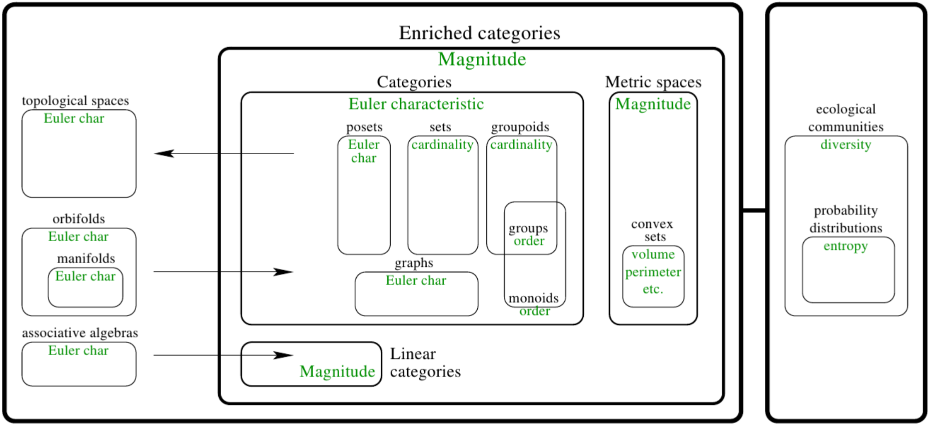
-  3 points
- 

area 3
-  dimension 3
-  3 holes

Very general question: what is the ‘size’ of a mathematical object?

(Category theory helps to make such questions precise.)

Overview of a project

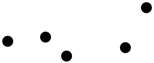
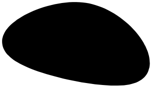


(just to indicate the scope. . .)

Metric spaces

A metric space is a collection of points with an assigned distance between each pair of points.

Examples

- 
- 
- the collection of all length-30 sequences like ATCCG...AGA
- a collection of species, with any sensible notion of distance between species.

The magnitude of a metric space

Every metric space has a magnitude, which is a real number measuring its ‘size’.

It can be thought of as the ‘effective number of points’.

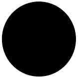
Examples

- $\mathrm{mag}(\bullet) = 1$
- $\mathrm{mag}(\bullet\!\!\bullet) = 1.01$ and $\mathrm{mag}(\bullet \qquad\qquad\qquad\qquad \bullet) = 1.99$
- $\mathrm{mag}\left(\bullet \qquad\qquad\qquad \begin{smallmatrix}\bullet \\ \bullet\end{smallmatrix} \right) = 2.01$ and $\mathrm{mag}\begin{pmatrix} \bullet & & \bullet \\ & & \\ & \bullet & \end{pmatrix} = 2.9.$

Magnitude was originally discovered by Solow and Polasky (1994) as a measure of diversity. They called it the ‘effective number of species’.

The magnitude of a metric space

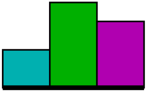
Magnitude appears to be closely related to classical geometric quantities:

Conjecture (with Simon Willerton, 2009) Let X be a convex set in the plane (e.g.  or  or  or ). Then

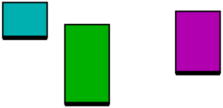
$$\mathrm{mag}(X) = \frac{1}{2\pi} \times \mathrm{area}(X) + \frac{1}{4} \times \mathrm{perimeter}(X) + 1.$$

How diversity fits in

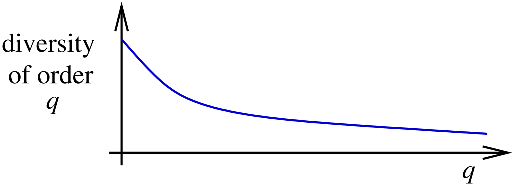
Recap of Christina's talk

Traditionally: diversity () = 2.4.

Taking into account the varying similarities between species:

diversity () = 2.1.

Actually, the diversity of a community isn't just a number, but a *family* of numbers:



The parameter q controls the emphasis placed on rare species.

These diversity measures satisfy various fundamental properties.

Maximizing diversity

Suppose we have a list of species, and we know how similar they are.

(Or in math-speak: suppose we have a metric space.)

Questions Which abundance distribution maximizes the diversity?
What is the value of the maximum diversity?

In principle, the answer depends on q.

Maximizing diversity

Theorem Neither depends on q. That is:

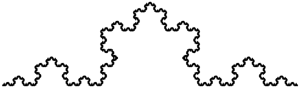
- There is a single abundance distribution that maximizes diversity of all orders q simultaneously.
- The value of the maximum diversity is the same for all q.

So each list of species (metric space) has an unambiguous maximum diversity $D_{\max}$.

When certain conditions are met, maximum diversity is equal to magnitude. (And it's *always* closely related.)

Applying biological concepts to pure mathematics

Every geometrical figure has a dimension:

-  has dimension 1
-  has dimension 2
-  has dimension 1.261...

Mark Meckes has used D_{max} to prove a pure-mathematical theorem on fractal dimension:

We end this section by considering a quantity related to magnitude which is in some ways better behaved. For a compact (not necessarily positive definite) metric space A, the **maximum diversity** of A is

$$|A|_+ = \sup_{\mu \in P(A)} \left(\int \int e^{-d(a,b)} \, d\mu(a) \, d\mu(b) \right)^{-1}, \tag{4.3}$$

where $P(A)$ denotes the space of Borel probability measures on A. By renormalization, this is simply what one obtains by restricting the supremum in (3.5) to positive measures; thus we trivially have

$$|A|_+ \leq |A| \tag{4.4}$$

for any compact PDMS A. The name stems from the following interpretation of the quantity

Summary

Diversity is one member of a large family of notions of ‘size’, extending across mathematics

Diversity is a fundamentally *mathematical* concept, not tied to any particular application

But thinking about applications has already advanced pure mathematics

Thanks

Christina Cobbold

Mark Meckes

Simon Willerton

You