The place of diversity in pure mathematics

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Mathematics in the early 20th century:
a ‘just so’ story
Pseudo-history

There are many things you can do in the plane... measure distance between points: \( \rightarrow \) metric spaces

**add vectors:**

\( x \quad y \quad x + y \)

\( \rightarrow \) groups

**scale:**

\( x \quad 1.5 x \)

\( \rightarrow \) vector spaces

**measure area:**

area of \( \rightarrow \) measure spaces

Realization It’s useful to study each aspect in isolation.

Example Consider all length-30 sequences of symbols A, G, C or T, such as

\[ \text{CGGATAACGTACCTCCAGGTTCACAAC} \]

We could define the distance between two sequences to be the number of places where they differ. This gives an example of a ‘metric space’.

(But we can’t add sequences together, or scale them, or measure the area of a set of sequences.)
Pseudo-history

Consequence of this approach: Mathematics split into many subdisciplines.

Danger: With many separate lines of development...

...we fail to see how they link up.

Category theory (my subject) provides a bird’s eye view:
How big is a thing?
Notions of size

There are many notions of size in mathematics:

- 3 points
- area 3
- dimension 3
- 3 holes

Very general question: what is the ‘size’ of a mathematical object? (Category theory helps to make such questions precise.)
Overview of a project

(just to indicate the scope... )
Metric spaces

A **metric space** is a collection of points with an assigned distance between each pair of points.

**Examples**

- the collection of all length-30 sequences like ATCCG... AGA
- a collection of species, with any sensible notion of distance between species.
The magnitude of a metric space

Every metric space has a magnitude, which is a real number measuring its ‘size’.

It can be thought of as the ‘effective number of points’.

Examples

- \( \text{mag}(\bullet) = 1 \)
- \( \text{mag}(\bullet\bullet) = 1.01 \) and \( \text{mag}(\bullet \bullet \bullet) = 1.99 \)
- \( \text{mag}(\bullet \bullet) = 2.01 \) and \( \text{mag} \begin{pmatrix} \bullet & \bullet \\ & \bullet \end{pmatrix} = 2.9. \)

Magnitude was originally discovered by Solow and Polasky (1994) as a measure of diversity. They called it the ‘effective number of species’.
The magnitude of a metric space

Magnitude appears to be closely related to classical geometric quantities:

Conjecture (with Simon Willerton, 2009) Let $X$ be a convex set in the plane (e.g. circle or square or triangle or ellipse). Then

$$
\text{mag}(X) = \frac{1}{2\pi} \times \text{area}(X) + \frac{1}{4} \times \text{perimeter}(X) + 1.
$$
How diversity fits in
Recap of Christina’s talk

Traditionally: diversity\(\left(\right) = 2.4.\)

Taking into account the varying similarities between species:

diversity\(\left(\right) = 2.1.\)

Actually, the diversity of a community isn’t just a number, but a family of numbers:

The parameter \(q\) controls the emphasis placed on rare species. These diversity measures satisfy various fundamental properties.
Maximizing diversity

Suppose we have a list of species, and we know how similar they are. (Or in math-speak: suppose we have a metric space.)

Questions Which abundance distribution maximizes the diversity? What is the value of the maximum diversity?

In principle, the answer depends on $q$. 
Maximizing diversity

**Theorem** Neither depends on $q$. That is:

- There is a single abundance distribution that maximizes diversity of all orders $q$ simultaneously.
- The value of the maximum diversity is the same for all $q$.

So each list of species (metric space) has an unambiguous maximum diversity $D_{\text{max}}$.

When certain conditions are met, maximum diversity is equal to magnitude. (And it’s *always* closely related.)
Applying biological concepts to pure mathematics

Every geometrical figure has a dimension:

• \( \text{has dimension } 1 \)

• \( \text{has dimension } 2 \)

• \( \text{has dimension } 1.261 \ldots \)

Mark Meckes has used \( D_{\text{max}} \) to prove a pure-mathematical theorem on fractal dimension:

\[
\sum_{\mathcal{P}(A)} |A|_+ = \sup_{\mu \in P(A)} \left( \int \int e^{-d(a,b)} d\mu(a) d\mu(b) \right)^{-1},
\]

where \( P(A) \) denotes the space of Borel probability measures on \( A \). By renormalization, this is simply what one obtains by restricting the supremum in (3.3) to positive measures; thus we trivially have

\[
|A|_+ \leq |A|
\]

for any compact PDMS \( A \). The name stems from the following interpretation of the quantity.
Summary
Diversity is one member of a large family of notions of ‘size’, extending across mathematics

Diversity is a fundamentally mathematical concept, not tied to any particular application

But thinking about applications has already advanced pure mathematics
Thanks

Christina Cobbold

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Simon Willerton

You