4.1.27 Let $\mathscr{A}$ be a locally small category, and let $A, A' \in \mathscr{A}$ with $H_A \cong H_{A'}$. Prove directly that $A \cong A'$.

4.1.28 Let p be a prime number. Show that the functor $U_p\colon \mathbf{Grp} \to \mathbf{Set}$ defined in Example 4.1.5 is isomorphic to $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$. (To check that there is an isomorphism of functors – that is, a *natural* isomorphism – you will first need to define U_p on maps. There is only one sensible way to do this.)

4.1.29 Using the result of Exercise 0.13(a), prove that the forgetful functor $\mathbf{CRing} \to \mathbf{Set}$ is isomorphic to $\mathbf{CRing}(\mathbb{Z}[x], -)$, as in Example 4.1.14.

4.1.30 The **Sierpiński space** is the two-point topological space S in which one of the singleton subsets is open but the other is not. Prove that for any topological space X, there is a canonical bijection between the open subsets of X and the continuous maps $X \to S$. Use this to show that the functor $\mathscr{O}$: $\mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ of Example 4.1.19 is represented by S.

4.1.31 Let $M\colon \mathbf{Cat} \to \mathbf{Set}$ be the functor that sends a small category $\mathscr{A}$ to the set of all maps in $\mathscr{A}$. Prove that M is representable.

4.1.32 Take locally small categories $\mathscr{A}$ and $\mathscr{B}$, and functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$. Show that F is left adjoint to G if and only if the two functors

$$\mathscr{B}(F(-), -),\ \mathscr{A}(-, G(-))\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Set}$$

of Remark 4.1.24 are naturally isomorphic. (Hint: this is made easier by using either Exercise 1.3.29 or Exercise 2.1.14.)

4.2 The Yoneda lemma

What do representables see?

Recall from Definition 1.2.15 that functors $\mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ are sometimes called 'presheaves' on $\mathscr{A}$. So for each $A \in \mathscr{A}$ we have a representable presheaf H_A, and we are asking how the rest of the presheaf category $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ looks from the viewpoint of H_A. In other words, if X is another presheaf, what are the maps $H_A \to X$?

Newcomers to category theory commonly find that the material presented in this section is where they first get stuck. Typically, the core of the difficulty is in understanding the question just asked. Let us ask it again.

We start by fixing a locally small category $\mathscr{A}$. We then take an object $A \in \mathscr{A}$ and a functor $X\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$. The object A gives rise to another functor

$H_A = \mathscr{A}(-, A)\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$. The question is: what are the maps $H_A \to X$? Since H_A and X are both objects of the presheaf category $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, the 'maps' concerned are maps in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$. So, we are asking what natural transformations

$$\mathscr{A}^{\mathrm{op}} \underset{X}{\overset{H_A}{\Downarrow}} \mathbf{Set} \tag{4.1}$$

there are. The set of such natural transformations is called

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X).$$

(This is a special case of the notation $\mathscr{B}(B, B')$ for the set of maps $B \to B'$ in a category $\mathscr{B}$. Here, $\mathscr{B} = [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, $B = H_A$, and $B' = X$.) We want to know what this set is.

There is an informal principle of general category theory that allows us to guess the answer. Look back at Remarks 1.1.2(b), 1.2.2(a) and 1.3.2(a) on the definitions of category, functor and natural transformation. Each remark is of the form 'from input of one type, it is possible to construct exactly one output of another type'. For example, in Remark 1.1.2(b), the input is a sequence of maps $A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n$, the output is a map $A_0 \to A_n$, and the statement is that no matter what we do with the input data $f_1, \ldots, f_n$, there is only one map $A_0 \to A_n$ that we can construct.

Let us apply this principle to our question. We have just seen how, given as input an object $A \in \mathscr{A}$ and a presheaf X on $\mathscr{A}$, we can construct a set, namely, $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$. Are there any other ways to construct a set from the same input data (A, X)? Yes: simply take the set $X(A)$! The informal principle suggests that these two sets are the same:

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \cong X(A) \tag{4.2}$$

for all $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$. This turns out to be true; and that is the Yoneda lemma.

Informally, then, the Yoneda lemma says that for any $A \in \mathscr{A}$ and presheaf X on $\mathscr{A}$:

A natural transformation $H_A \to X$ is an element of $X(A)$.

Here is the formal statement. The proof follows shortly.

Theorem 4.2.1 (Yoneda) *Let $\mathscr{A}$ be a locally small category. Then*

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \cong X(A) \tag{4.3}$$

naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.

This is exactly what was stated in (4.2), except that the word 'naturally' has appeared. Recall from Definition 1.3.12 that for functors $F, G\colon \mathscr{C} \to \mathscr{D}$, the phrase '$F(C) \cong G(C)$ naturally in C' means that there is a natural isomorphism $F \cong G$. So the use of this phrase in the Yoneda lemma suggests that each side of (4.3) is functorial in both A and X. This means, for instance, that a map $X \to X'$ must induce a map

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X'),$$

and that not only does the isomorphism (4.3) hold for *every* A and X, but also, the isomorphisms can be chosen in a way that is compatible with these induced maps. Precisely, the Yoneda lemma states that the composite functor

$$\begin{array}{ccccc} \mathscr{A}^{\mathrm{op}} \times [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] & \xrightarrow{H_\bullet^{\mathrm{op}} \times 1} & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}} \times [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] & \xrightarrow{\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]}} & \mathbf{Set} \\ (A, X) & \longmapsto & (H_A, X) & \longmapsto & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \end{array}$$

is naturally isomorphic to the evaluation functor

$$\begin{array}{ccc} \mathscr{A}^{\mathrm{op}} \times [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] & \to & \mathbf{Set} \\ (A, X) & \mapsto & X(A). \end{array}$$

If the Yoneda lemma were false then the world would look much more complex. For take a presheaf $X\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$, and define a new presheaf X' by

$$X' = [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_\bullet, X)\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set},$$

that is, $X'(A) = [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$ for all $A \in \mathscr{A}$. Yoneda tells us that $X'(A) \cong X(A)$ naturally in A; in other words, $X' \cong X$. If Yoneda were false then starting from a single presheaf X, we could build an infinite sequence $X, X', X'', \ldots$ of new presheaves, potentially all different. But in reality, the situation is very simple: they are all the same.

The proof of the Yoneda lemma is the longest proof so far. Nevertheless, there is essentially only one way to proceed at each stage. If you suspect that you are one of those newcomers to category theory for whom the Yoneda lemma presents the first serious challenge, an excellent exercise is to work out the proof before reading it. No ingenuity is required, only an understanding of all the terms in the statement.

Proof of the Yoneda lemma We have to define, for each A and X, a bijection between the sets $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$ and $X(A)$. We then have to show that our bijection is natural in A and X.

First, fix $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$. We define functions

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \underset{(\tilde{\ })}{\overset{(\hat{\ })}{\rightleftarrows}} X(A) \tag{4.4}$$

and show that they are mutually inverse. So we have to do four things: define the function $(\hat{\ })$, define the function $(\tilde{\ })$, show that $(\hat{\tilde{\ }})$ is the identity, and show that $(\tilde{\hat{\ }})$ is the identity.

- Given $\alpha \colon H_A \to X$, define $\hat{\alpha} \in X(A)$ by $\hat{\alpha} = \alpha_A(1_A)$. (How else could we possibly define it?)
- Let $x \in X(A)$. We have to define a natural transformation $\tilde{x} \colon H_A \to X$. That is, we have to define for each $B \in \mathscr{A}$ a function

  $$\tilde{x}_B \colon H_A(B) = \mathscr{A}(B, A) \to X(B)$$

  and show that the family $\tilde{x} = (\tilde{x}_B)_{B \in \mathscr{A}}$ satisfies naturality.

  Given $B \in \mathscr{A}$ and $f \in \mathscr{A}(B, A)$, define

  $$\tilde{x}_B(f) = (X(f))(x) \in X(B).$$

  (How else could we possibly define it?) This makes sense, since $X(f)$ is a map $X(A) \to X(B)$. To prove naturality, we must show that for any map $B' \xrightarrow{g} B$ in $\mathscr{A}$, the square

  $$\begin{array}{ccc} \mathscr{A}(B, A) & \xrightarrow{H_A(g) = - \circ g} & \mathscr{A}(B', A) \\ {\scriptstyle \tilde{x}_B} \downarrow & & \downarrow {\scriptstyle \tilde{x}_{B'}} \\ X(B) & \xrightarrow[X(g)]{} & X(A) \end{array}$$

  commutes. To reduce clutter, let us write $X(g)$ as Xg, and so on. Now for all $f \in \mathscr{A}(B, A)$, we have

  $$\begin{array}{ccc} f & \longmapsto & f \circ g \\ \downarrow & & \downarrow \\ & & (X(f \circ g))(x) \\ (Xf)(x) & \longmapsto & (Xg)((Xf)(x)), \end{array}$$

  and $X(f \circ g) = (Xg) \circ (Xf)$ by functoriality, so the square does commute.
- Given $x \in X(A)$, we have to show that $\hat{\tilde{x}} = x$, and indeed,

  $$\hat{\tilde{x}} = \tilde{x}_A(1_A) = (X1_A)(x) = 1_{X(A)}(x) = x.$$

- Given $\alpha\colon H_A \to X$, we have to show that $\tilde{\hat{\alpha}} = \alpha$. Two natural transformations are equal if and only if all their components are equal; so, we have to show that $\left(\tilde{\hat{\alpha}}\right)_B = \alpha_B$ for all $B \in \mathscr{A}$. Each side of this equation is a function from $H_A(B) = \mathscr{A}(B, A)$ to $X(B)$, and two functions are equal if and only if they take equal values at every element of the domain; so, we have to show that

  $$\left(\tilde{\hat{\alpha}}\right)_B(f) = \alpha_B(f)$$

  for all $B \in \mathscr{A}$ and $f\colon B \to A$ in $\mathscr{A}$. The left-hand side is by definition

  $$\left(\tilde{\hat{\alpha}}\right)_B(f) = (Xf)(\hat{\alpha}) = (Xf)(\alpha_A(1_A)),$$

  so it remains to prove that

  $$(Xf)(\alpha_A(1_A)) = \alpha_B(f). \tag{4.5}$$

  By naturality of α (the only tool at our disposal), the square

  $$\begin{array}{ccc} \mathscr{A}(A, A) & \xrightarrow{H_A(f) = - \circ f} & \mathscr{A}(B, A) \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ X(A) & \xrightarrow[Xf]{} & X(B) \end{array}$$

  commutes, which when taken at $1_A \in \mathscr{A}(A, A)$ gives equation (4.5).

(The proof is not over yet, but it is worth pausing to consider the significance of the fact that $\tilde{\hat{\alpha}} = \alpha$. Since $\hat{\alpha}$ is the value of α at 1_A, this implies:

A natural transformation $H_A \to X$ is determined by its value at 1_A.

Just *how* a natural transformation $H_A \to X$ is determined by its value at 1_A is described in equation (4.5).)

This establishes the bijection (4.4) for each $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$. We now show that the bijection is natural in A and X.

We employ two mildly labour-saving devices. First, in principle we have to prove naturality of both $\hat{(\ )}$ and $\tilde{(\ )}$, but by Lemma 1.3.11, it is enough to prove naturality of just one of them. We prove naturality of $\hat{(\ )}$. Second, by Exercise 1.3.29, $\hat{(\ )}$ is natural in the pair (A, X) if and only if it is natural in A for each fixed X and natural in X for each fixed A. So, it remains to check these two types of naturality.

Naturality in A states that for each $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ and $B \xrightarrow{f} A$ in $\mathscr{A}$, the square

$$
\begin{array}{ccc}
[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) & \xrightarrow{-\circ H_f} & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_B, X) \\
\Big\downarrow {\scriptstyle (\hat{\ })} & & \Big\downarrow {\scriptstyle (\hat{\ })} \\
X(A) & \xrightarrow[Xf]{} & X(B)
\end{array}
$$

commutes. For $\alpha \colon H_A \to X$, we have

$$
\begin{array}{ccc}
\alpha & \longmapsto & \alpha \circ H_f \\
\big\downarrow & & \big\downarrow \\
 & & (\alpha \circ H_f)_B(1_B) \\
\alpha_A(1_A) & \longmapsto & (Xf)(\alpha_A(1_A)),
\end{array}
$$

so we have to show that $(\alpha \circ H_f)_B(1_B) = (Xf)(\alpha_A(1_A))$. Indeed,

$$
\begin{aligned}
(\alpha \circ H_f)_B(1_B) &= \alpha_B((H_f)_B(1_B)) \\
&= \alpha_B(f \circ 1_B) = \alpha_B(f) \\
&= (Xf)(\alpha_A(1_A)),
\end{aligned}
$$

where the first step is by definition of composition in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, the second is by definition of H_f, and the last is by equation (4.5).

Naturality in X states that for each $A \in \mathscr{A}$ and map

$$
\mathscr{A}^{\mathrm{op}} \underset{X'}{\overset{X}{\Downarrow \theta}} \mathbf{Set}
$$

in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, the square

$$
\begin{array}{ccc}
[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) & \xrightarrow{\theta\circ -} & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X') \\
\Big\downarrow {\scriptstyle (\hat{\ })} & & \Big\downarrow {\scriptstyle (\hat{\ })} \\
X(A) & \xrightarrow[\theta_A]{} & X'(A)
\end{array}
$$

commutes. For $\alpha \colon H_A \to X$, we have

$$
\begin{array}{ccc}
\alpha & \longmapsto & \theta \circ \alpha \\
\big\downarrow & & \big\downarrow \\
 & & (\theta \circ \alpha)_A(1_A) \\
\alpha_A(1_A) & \longmapsto & \theta_A(\alpha_A(1_A)),
\end{array}
$$

and $(\theta \circ \alpha)_A = \theta_A \circ \alpha_A$ by definition of composition in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, so the square does commute. This completes the proof. □

Exercises

4.2.2 State the dual of the Yoneda lemma.

4.2.3 One way to understand the Yoneda lemma is to examine some special cases. Here we consider one-object categories.

Let M be a monoid. The underlying set of M can be given a right M-action by multiplication: $x \cdot m = xm$ for all $x, m \in M$. This M-set is called the **right regular representation** of M. Let us write it as $\underline{M}$.

(a) When M is regarded as a one-object category, functors $M^{\mathrm{op}} \to \mathbf{Set}$ correspond to right M-sets (Example 1.2.14). Show that the M-set corresponding to the unique representable functor $M^{\mathrm{op}} \to \mathbf{Set}$ is the right regular representation.
(b) Now let X be any right M-set. Show that for each $x \in X$, there is a unique map $\alpha\colon \underline{M} \to X$ of right M-sets such that $\alpha(1) = x$. Deduce that there is a bijection between {maps $\underline{M} \to X$ of right M-sets} and X.
(c) Deduce the Yoneda lemma for one-object categories.

4.3 Consequences of the Yoneda lemma

The Yoneda lemma is fundamental in category theory. Here we look at three important consequences.

Notation 4.3.1 An arrow decorated with a $\sim$, as in $A \xrightarrow{\sim} B$, denotes an isomorphism.

A representation is a universal element

Corollary 4.3.2 *Let $\mathscr{A}$ be a locally small category and $X\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$. Then a representation of X consists of an object $A \in \mathscr{A}$ together with an element $u \in X(A)$ such that:*

$$\text{for each } B \in \mathscr{A} \text{ and } x \in X(B) \text{, there is a unique map } \bar{x}\colon B \to A \text{ such that } (X\bar{x})(u) = x. \tag{4.6}$$

To clarify the statement, first recall that by definition, a representation of X is an object $A \in \mathscr{A}$ together with a natural isomorphism $\alpha\colon H_A \xrightarrow{\sim} X$. Corollary 4.3.2 states that such pairs (A, α) are in natural bijection with pairs (A, u) satisfying condition (4.6).

Pairs (B, x) with $B \in \mathscr{A}$ and $x \in X(B)$ are sometimes called **elements** of the presheaf X. (Indeed, the Yoneda lemma tells us that x amounts to a generalized element of X of shape H_B.) An element u satisfying condition (4.6)