The many faces of magnitude

Tom Leinster

Edinburgh

Aim of this talk

Connect up various quantities in mathematics that might be called ‘size’, e.g.

- cardinality
- measure
- Euler characteristic
- entropy

Plan:

1. Matrices
2. Categories and topological spaces
3. Algebras
4. Metric spaces
5. Graphs
6. Diversity

1. Matrices

The magnitude of a matrix

Let Z be a matrix.

A weighting on Z is a column vector w such that

$$Zw = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Suppose that both Z and Z^{t} have at least one weighting.

The magnitude of Z is

$$|Z| = \sum_i w_i$$

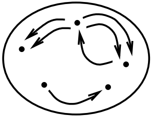
for any weighting w. This is independent of choice of w.

E.g.: If Z is invertible then $|Z| = \sum_{i,j} (Z^{-1})_{ij}$.

2. Categories and topological spaces

The magnitude of a category

Let $\mathbf{C}$ be a finite category with objects $c_1, \ldots, c_n$.



Write $Z_{\mathbf{C}}$ for the $n \times n$ matrix with (i,j)-entry $|\mathrm{Hom}(c_i, c_j)|$.

The magnitude of $\mathbf{C}$ is $|\mathbf{C}| = |Z_{\mathbf{C}}| \in \mathbb{Q}$.

E.g.: $|\bullet \quad \bullet \quad \bullet| = \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 3.$

E.g.: Let $\mathbf{C} = \left(\bullet \rightrightarrows \bullet \right)$. Then

$$Z_{\mathbf{C}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad Z_{\mathbf{C}}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

so

$$|\mathbf{C}| = 1 + -2 + 0 + 1 = 0 = \chi(S^1).$$

The Euler characteristic of a space

Every (small) category $\mathbf{C}$ gives rise to a topological space $B\mathbf{C}$,
its classifying space.

E.g.: If $\mathbf{C} = (\bullet \quad \bullet \quad \bullet)$ then $B\mathbf{C} = (\bullet \quad \bullet \quad \bullet)$, discrete 3-point space.

E.g.: If $\mathbf{C} = \left(\bullet \rightrightarrows \bullet \right)$ then $B\mathbf{C} = S^1$.

Theorem: *Let* $\mathbf{C}$ *be a finite category containing no nontrivial isomorphisms or endomorphisms. Then* $|\mathbf{C}| = \chi(B\mathbf{C})$.

Given a triangulated manifold M, the simplices form a partially ordered set $\mathbf{C}_M$, which can be viewed as a category.

Theorem: *Let M be a compact triangulated manifold. Then* $\chi(M) = |\mathbf{C}_M|$.

Rough conclusion: The notions of (i) Euler characteristic of a space and (ii) magnitude of a category can each be defined in terms of the other.

3. Algebras

The Euler characteristic of an algebra

Let k be a field and A an associative algebra over k.

Suppose k is algebraically closed and A is Koszul, of finite dimension and global dimension.

Up to isomorphism, there are only finitely many projective indecomposable A-modules. Call them $M_1, \ldots, M_n$.

Write Z_A for the $n \times n$ matrix with (i, j)-entry $\dim(\mathrm{Hom}_A(M_i, M_j))$.

The magnitude of A is $|A| = |Z_A|$.

Theorem (with Catharina Stroppel)

$$|A| = \sum_{r=0}^{\infty} (-1)^r \dim(\mathrm{Ext}_A^r(A_0, A_0)).$$

4. Metric spaces

The magnitude of a finite metric space

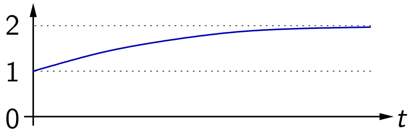
Let $A = \{a_1, \ldots, a_n\}$ be a finite metric space.

Write Z_A for the $n \times n$ matrix with (i, j)-entry $e^{-d(a_i, a_j)}$.

The magnitude of A is $|A| = |Z_A| \in \mathbb{R}$.

Examples

- $|\emptyset| = 0$ and $|\bullet| = 1$.
- $\left|\bullet \overset{\leftarrow\, t\, \rightarrow}{} \bullet\right| = \frac{2}{1+e^{-t}}$:



- If $d(a_i, a_j) = \infty$ for all $i \neq j$ then $|A| = n$.

Slogan: Magnitude is the 'effective number of points'.

The magnitude of a compact metric space

Theorem (Mark Meckes)

All sensible ways of extending the definition of magnitude from finite spaces to compact spaces give the same answer, at least for 'good' spaces (e.g. compact subspaces of $\mathbb{R}^N$).

E.g.: $|[0,\ell]| = 1 + \ell/2$. So, *the magnitude of a line tells you its length*.

Let $A \subseteq \mathbb{R}^N$ be a compact space.
Given $t > 0$, write tA for A scaled by a factor of t.

The magnitude function of A is the function

$$\begin{array}{ccc} (0,\infty) & \longrightarrow & \mathbb{R}, \\ t & \mapsto & |tA|. \end{array}$$

E.g.: The magnitude function of $[0,\ell]$ is $t \mapsto 1 + (\ell/2) \cdot t$.

Geometric measures of a convex set

Conjecture (with Simon Willerton)

For compact convex $A \subseteq \mathbb{R}^2$,

$$|A| = \chi(A) + \tfrac{1}{4}\,\mathrm{perimeter}(A) + \tfrac{1}{2\pi}\,\mathrm{area}(A).$$

Equivalently: for compact convex $A \subseteq \mathbb{R}^2$ and $t > 0$,

$$|tA| = \chi(A) + \tfrac{1}{4}\,\mathrm{perimeter}(A) \cdot t + \tfrac{1}{2\pi}\,\mathrm{area}(A) \cdot t^2.$$

So:

the magnitude function of a convex planar set
tells you its Euler characteristic, perimeter and area.

Moreover, the degree of the polynomial is the dimension of the space.

5. Graphs

The magnitude of a graph

Graph will mean finite, undirected graph with no multiple edges or loops.

Let G be a graph, with vertices $a_1, \ldots, a_n$.

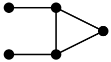
The distance between two vertices is the shortest path-length between them.

Let Z_G be the $n \times n$ matrix with (i, j)-entry $q^{d(a_i, a_j)}$, where q is a formal variable (and $q^\infty = 0$).

Then Z_G is invertible over the field $\mathbb{Q}(q)$ of rational functions in q.

The magnitude of G is $|G| = |Z_G| \in \mathbb{Q}(q)$.

E.g.:

$\Bigg|$  $\Bigg| = \dfrac{5 + 5q - 4q^2}{(1+q)(1+2q)}.$

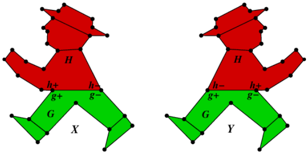
The magnitude of a graph: properties

Cardinality-like properties

- $|\bullet \quad \bullet \quad \bullet| = 3$, etc.
- $|G \,\square\, H| = |G| \cdot |H|$, where $\square$ is the 'cartesian product' of graphs
- $|G \cup H| = |G| + |H| - |G \cap H|$, under hypotheses.

Magnitude also bears some resemblance to the Tutte polynomial.

For instance, the two graphs



have the same magnitude.

But neither magnitude nor the Tutte polynomial is determined by the other.

6. Diversity

joint with Christina Cobbold

A spectrum of viewpoints on biodiversity

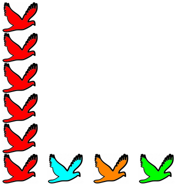
Conserving *species*
is what matters

Conserving *communities*
is what matters

Rare species
count for as much
as common ones
—every species is precious

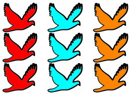
Common species
are the really
important ones
—they shape the community

This ⟶ ⟵ This



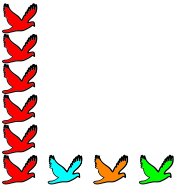
is more diverse than is less diverse than

that ⟶ ⟵ that

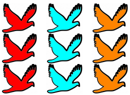


A spectrum of viewpoints on biodiversity

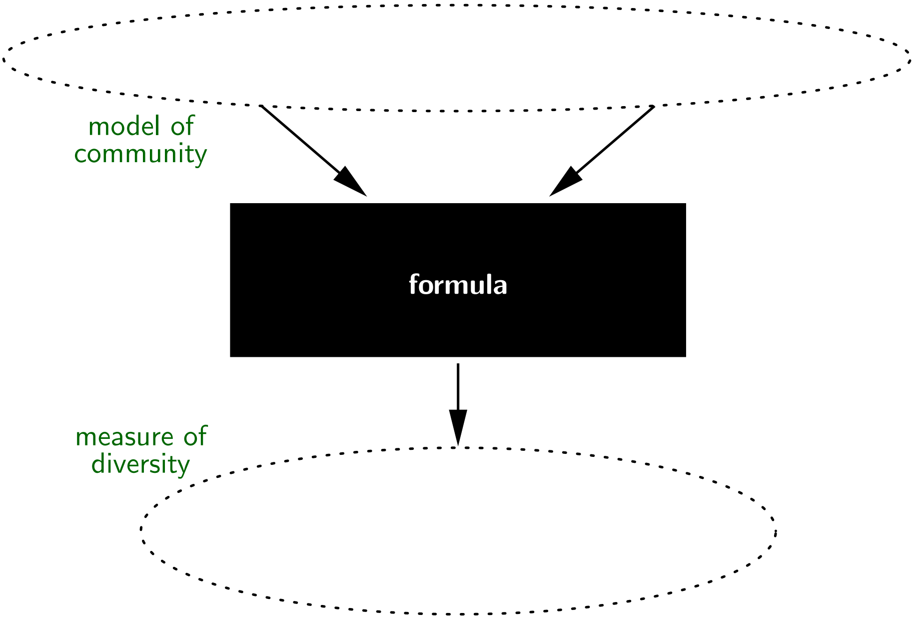
Rare species are important —— *Rare species are unimportant*

This ⟶  ⟵ This

is more diverse than | is less diverse than

that ⟶  ⟵ that

Quantifying diversity



Quantifying diversity

similarity matrix Z

model of community

Quantifying diversity

similarity matrix Z

$n \times n$ matrix (n = number of species)

Z_{ij} = similarity between ith and jth species = Z_{ji}

$0 \leq Z_{ij} \leq 1$ and $Z_{ii} = 1$

(0: totally dissimilar; 1: identical)

E.g.: Naive model: $Z = I$ (different species have *nothing* in common).

E.g.: Genetic similarity.

E.g.: Taxonomic: e.g. $Z_{ij} = \begin{cases} 1 & \text{if same species} \\ 0.7 & \text{if different species but same genus} \\ 0 & \text{otherwise.} \end{cases}$

Quantifying diversity

similarity matrix Z

frequency distribution p

model of community

Quantifying diversity

similarity matrix Z

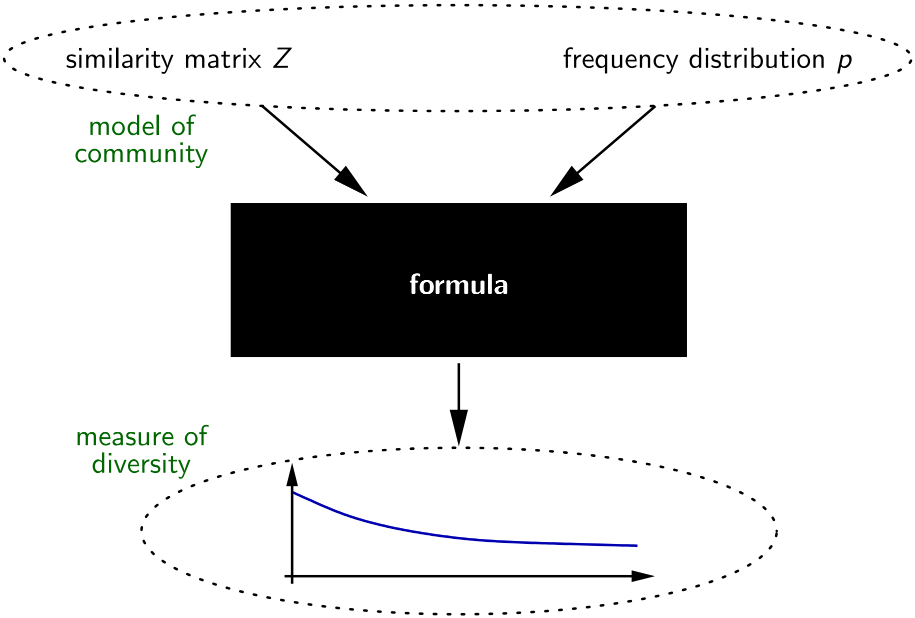
frequency distribution p

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

p_i = relative frequency, or relative abundance, of the ith species

$p_i \geq 0$ and $\sum p_i = 1$

Quantifying diversity



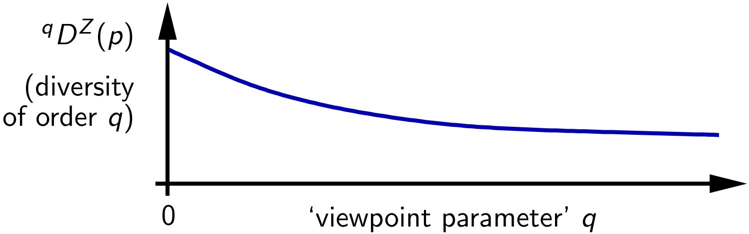
Quantifying diversity

similarity matrix Z

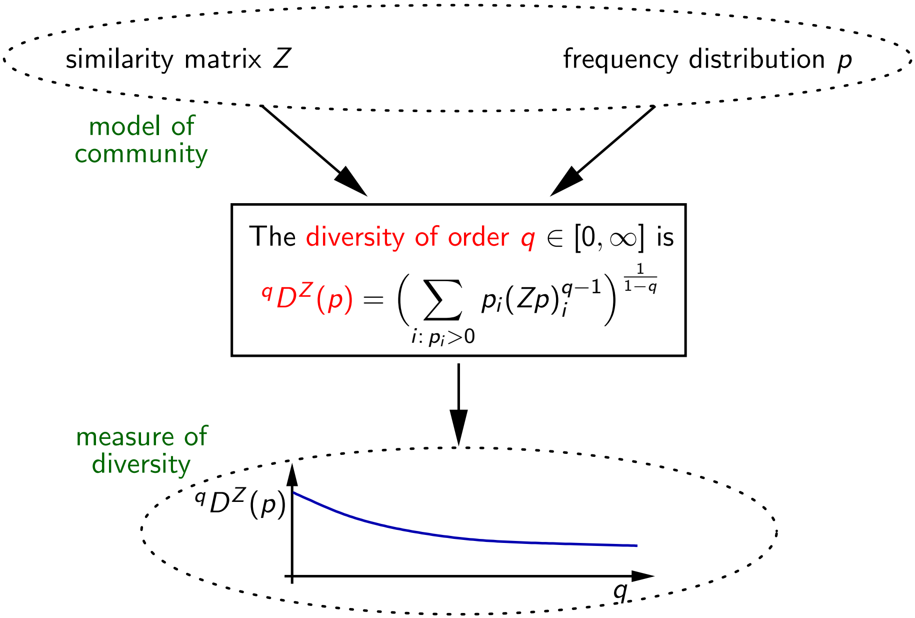
frequency distribution p

$q = 0$:
rare species are important
↓

$q = \infty$:
rare species are unimportant
↓



Quantifying diversity



The maximum diversity problem

Fix a list of n species, with similarity matrix Z.

Problem: By choosing p intelligently, how large can we make the diversity?

More exactly: Let $0 \leq q \leq \infty$.

What's the maximum diversity of order q, and which distributions attain it?

Theorem

1. *There is a distribution that maximizes diversity of all orders at once.*
2. *The maximum diversity of order q is independent of q. Call it $D_{max}(Z)$.*
3. *$D_{max}(Z)$ is closely related to (and often equal to) $|Z|$.*

E.g.: For taxonomic similarity matrices as above, $D_{\max}(Z) = |Z|$.

Moral: magnitude $\approx$ maximum diversity.

Postscript: Landscape ecology

A fundamental question

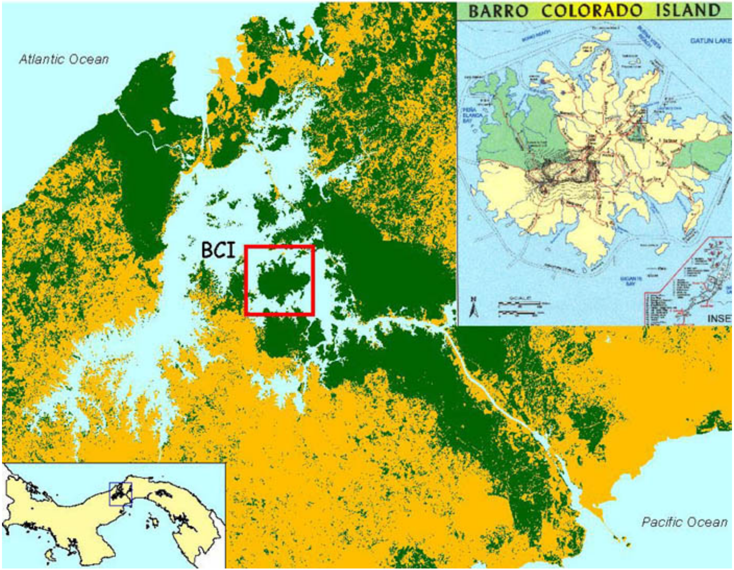
Money for conservation is scarce.

How do we decide where to spend it?

We need quantitative tools to identify areas of high diversity or high interest.

Building on our work, Richard Reeve and Louise Matthews have developed some such tools. . .

Barro Colorado Island, Panama

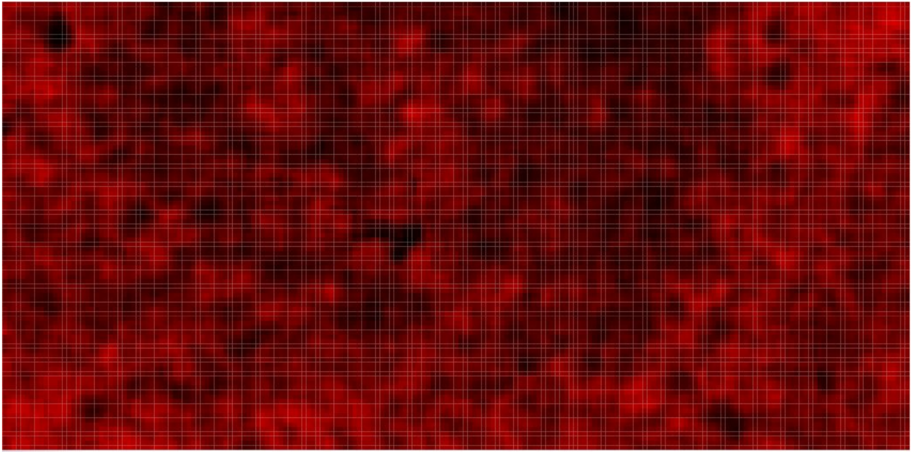


The island is mostly tropical forest.

Barro Colorado Island, Panama

Red: areas of high variation.

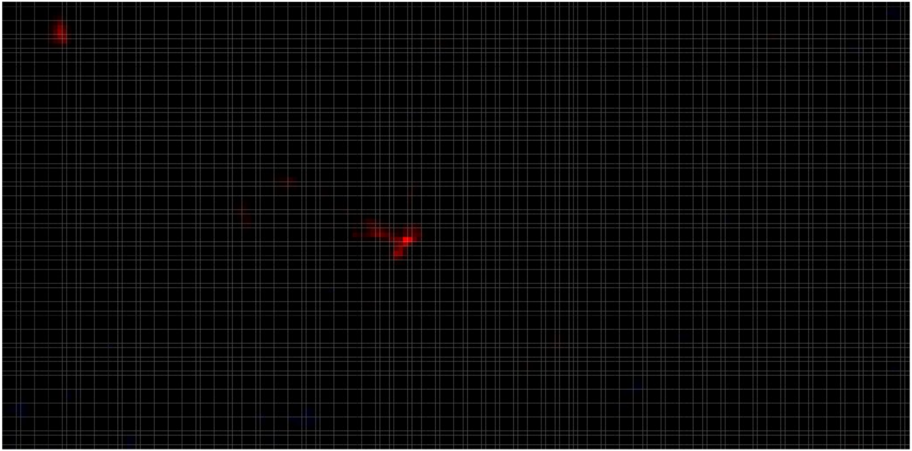
Black: areas of low variation.



Barro Colorado Island, Panama

Red: areas most different from rest of forest.

Black: areas most similar to rest of forest.



Thanks


Christina Cobbold


Louise Matthews


Mark Meckes


Richard Reeve


Catharina Stroppel


Simon Willerton

The Barro Colorado Island project

(The BCI forest dynamics research project was made possible by National Science Foundation grants to Stephen P. Hubbell: DEB-0640386, DEB-0425651, DEB- 0346488, DEB-0129874, DEB-00753102, DEB-9909347, DEB-9615226, DEB- 9615226, DEB-9405933, DEB-9221033, DEB-9100058, DEB-8906869, DEB- 8605042, DEB-8206992, DEB-7922197, support from the Center for Tropical Forest Science, the Smithsonian Tropical Research Institute, the John D. and Catherine T. MacArthur Foundation, the Mellon Foundation, the Small World Institute Fund, and numerous private individuals, and through the hard work of over 100 people from 10 countries over the past two decades. The plot project is part the Center for Tropical Forest Science, a global network of large-scale demographic tree plots.)