The magnitude of metric spaces I

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Parts joint with

Mark Meckes (Case Western)
Simon Willerton (Sheffield)

These slides are available on my web page
Background
Cardinality-like invariants

For many mathematical objects, there is a canonical notion of size.
Cardinality-like invariants

For many mathematical objects, there is a canonical notion of size.

Sets have cardinality

Vector spaces have dimension

Topological spaces have Euler characteristic

Posets have Euler characteristic

Probability spaces have entropy

Purpose of talk: introduce a new canonical notion of size...

Metric spaces have magnitude... and provide evidence that it subsumes many invariants of integral geometry.
Cardinality-like invariants

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... and provide evidence that it subsumes many invariants of integral geometry.
Where does magnitude come from?

There is a general concept of enriched category. It includes:

• sets
• posets
• (ordinary) categories
• associative algebras
• metric spaces
• . . .

There is a general definition of the magnitude of a (suitably finite) enriched category. It includes:

• cardinality of finite sets
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2. Magnitude of infinite metric spaces

Tomorrow (Simon Willerton)
Example computations
Asymptotic behaviour of magnitude
Magnitude of manifolds
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Magnitude of manifolds
1. Magnitude of finite metric spaces
The definition of magnitude

Let \( A = \{a_1, \ldots, a_n\} \) be a finite metric space. Write \( Z_A \) for the \( n \times n \) matrix with \( (Z_A)_{ij} = e^{-d(a_i, a_j)} \in [0,1] \).

A weighting on \( A \) is a column vector \( w \) such that \( Z_A w = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \).

If \( A \) admits a weighting, the magnitude of \( A \) is \( |A| = w_1 + \cdots + w_n \).

Fact: This is independent of the choice of weighting.

'Usually' \( Z_A \) is invertible. Then there is exactly one weighting, and \( |A| = n \sum_{i,j=1}^n \left( Z_A^{-1} \right)_{ij} \).
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Let $A = \{a_1, \ldots, a_n\}$ be a finite metric space.

[continued on next page]
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$$|A| = \sum_{i,j=1}^{n} (Z_A^{-1})_{ij}.$$
Basic examples

- $|\emptyset| = 0$ and $|\bullet| = 1$

- Let $A = (\bullet \leftarrow r \rightarrow \bullet)$. Then $Z_A = (e^{-0} e^{-r} e^{-r} e^{-0}) = (1 e^{-r} e^{-r} 1)$

- $|A| = \text{sum of all four entries of } Z_A - 1 = 1 + \tanh(r/2)$.

- If $d(a, b) = \infty$ for all $a \neq b$ then $|A| = \#A$: magnitude = cardinality.
Basic examples

- \(|\emptyset| = 0\)

- Let \(A = (\bullet \leftarrow r \rightarrow \bullet)\). Then \(Z_A = (e^{-0} e^{-r} e^{-r} e^{-0}) = (1 e^{-r} e^{-r} 1)\) and \(|A| = \text{sum of all four entries of } Z - 1\). If \(d(a, b) = \infty\) for all \(a \neq b\) then \(|A| = |A|: \text{magnitude} = \text{cardinality} \).
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• If $d(a, b) = \infty$ for all $a \neq b$ then $|A| = \#A$: magnitude = cardinality.
The magnitude function of a space

Magnitude assigns to each metric space not just a number, but a function. For \( t > 0 \), write \( tA \) for \( A \) scaled up by a factor of \( t \):

\[
d_{tA}(a, b) = t d(a, b).
\]

The magnitude function of a metric space \( A \) is the partially-defined function \((0, \infty) \to \mathbb{R} \) \( t \mapsto |tA| \). E.g.:

The magnitude function of \( A = (\bullet ← 1 → \bullet) \) is

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\begin{align*}
|tA| & = 0 & \quad & t < 1 \\
& = 1 + \tanh\left(\frac{t}{2}\right) & \quad & t \geq 1
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Magnitude assigns to each metric space not just a *number*, but a *function*. 

For $t > 0$, write $tA$ for $A$ scaled up by a factor of $t$:

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$$1 + \tanh(t/2)$$
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The magnitude function of $A$ is the partially-defined function

$$(0, \infty) \rightarrow \mathbb{R}$$

$t \mapsto |tA|.$

Theorem

Let $A$ be a finite metric space.

Then:

• The magnitude function of $A$ has only finitely many singularities
• For $t \gg 0$, the magnitude function of $A$ is strictly increasing
• $\lim_{t \to \infty} |tA| = \#A.$
The magnitude function of a space

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Let $A$ be a finite metric space.
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The magnitude function of A is the partially-defined function

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Warning example:
Let $A$ be the 5-point space given by the shortest-path metric on the graph opposite.
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![Graph](graph.png)
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- $|tA| > \#A$
- $|tA|$ decreasing
- $|tA|$ undefined
- $|tA| < 0$; $\emptyset \subset tA$ but $|\emptyset| > |tA|$
Positive definite spaces

Roughly, these are the spaces for which ‘surprising’ behaviour does \textit{not} occur.
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Definition

A finite metric space $A$ is positive definite if its matrix $Z_A$ is positive definite.
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Let $A = \{a_1, \ldots, a_n\}$ be a positive definite metric space.
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- $|A| \geq 0$
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Let $A = \{a_1, \ldots, a_n\}$ be a positive definite metric space. Then:

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- $|A| \geq 0$
- every subspace $B \subseteq A$ is positive definite, and $|B| \leq |A|$
- $|A| = \sup_{v \in \mathbb{R}^n \setminus \{0\}} \frac{(\sum v_i)^2}{v^t Z_A v}$. 
Subsets of $\mathbb{R}^N$

Theorem
Every finite subset of $\mathbb{R}^N$ is positive definite.

In particular, every finite subset of $\mathbb{R}^N$ has well-defined magnitude.

Outline of proof:

• Reduce to showing that the Fourier transform of $x \mapsto e^{-\|x\|}$ is everywhere positive.
• Use known formula for this Fourier transform.

More generally, write $\ell^N_p$ for $\mathbb{R}^N$ with the $\ell^p$ metric.

Theorem (Meckes)
Let $p \leq 2$. Then every finite subset of $\ell^N_p$ is positive definite.
Subsets of $\mathbb{R}^N$

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Theorem (Meckes)

Let $p \leq 2$. Then every finite subset of $\ell^N_p$ is positive definite.
Digression: diversity and entropy

There is a definition of the entropy of a probability distribution on a finite set. There is also a definition of the entropy of a probability distribution on a finite metric space, taking the metric into account. This is important in theoretical ecology:

- points represent species
- distances represent differences (e.g. genetic) between species
- probabilities represent relative frequencies of species
- entropy measures biological diversity.

Maximum diversity/entropy problem:

Given a list of species, which frequency distribution maximizes the diversity?

The solution is given in terms of weightings and magnitude. Magnitude can be understood as something like maximum entropy.
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2. Magnitude of infinite metric spaces
Idea: Define magnitude of infinite spaces via finite approximations. This works best if we stay in the world of positive definite spaces.

Definition: A metric space is positive definite if every finite subspace is positive definite.

Example: $\mathbb{R}^N$ is positive definite.

Definition: Let $A$ be a compact, positive definite metric space. The magnitude of $A$ is $|A| = \sup\{|B|: B \text{ is a finite subset of } A\} \in [0, \infty]$. (These definitions are consistent with the definitions for finite spaces.)
From finite to infinite spaces

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From finite to infinite spaces (digression)

Alternative idea: Instead of using finite approximations, work directly with measures on the space.

A weight measure on a compact metric space $A$ is a signed Borel measure $w$ such that

$$\int_{A} e^{-d(a,b)} \, dw(b) = 1.$$ 

If a weight measure exists, the measure magnitude of $A$ is $w(A)$.

Meckes has theorems stating that the two approaches give the same answers, in so far as measure magnitude is defined.

But the measure approach currently has some limitations. So in what follows, we use the finite-approximation definition of magnitude.
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Theorem

Let \( L \geq 0 \). Let \((A_k)\) be a sequence of finite subsets of \( \mathbb{R} \) such that

\[
\lim_{k \to \infty} A_k = [0, L]
\]

in the Hausdorff topology.

Then

\[
\lim_{k \to \infty} |A_k| = 1 + \frac{1}{2}L.
\]

Hence

\[
|[0, L]| = 1 + \frac{1}{2}L,
\]

and \([0, L]\) has magnitude function \( t \mapsto |[0, tL]| = |[0, tL]| = 1 + \frac{1}{2}L \cdot t \).

Magnitude comes from enriched category theory. . . but produces geometric invariants.
Line segments

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Euler characteristic
dimension
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Magnitude comes from enriched category theory...
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Magnitude comes from enriched category theory... but produces geometric invariants.
Let $A$ and $B$ be metric spaces. Write $A \otimes B$ for their '$\ell^1$-product': the set of points is $A \times B$, and $d_{A \otimes B}((a, b), (a', b')) = d_A(a, a') + d_B(b, b')$.

E.g.: a cuboid $[0, L_1] \times \cdots \times [0, L_N] \subset \ell^N$, with the subspace metric, is $[0, L_1] \otimes \cdots \otimes [0, L_N]$ as an abstract metric space.
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**Lemma**

$|A \otimes B| = |A| \cdot |B|$. 
Cuboids

Can now calculate magnitude function of $[0, L_1] \times [0, L_2] \subset \ell^2_1$.
Cuboids

Can now calculate magnitude function of \([0, L_1] \times [0, L_2] \subset \ell^2_1\): it is

\[
t \mapsto |t([0, L_1] \otimes [0, L_2])|
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Cuboids

Can now calculate magnitude function of $[0, L_1] \times [0, L_2] \subset \ell_1^2$: it is

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= (1 + \frac{1}{2}L_1 t) \cdot (1 + \frac{1}{2}L_2 t)
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Cuboids

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Euler characteristic
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Euler characteristic, semiperimeter, area
**Cuboids**

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Euler characteristic  semiperimeter  area  dimension
Cuboids

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In general, the magnitude function of the cuboid

$A = [0, L_1] \times \cdots \times [0, L_N] \subset \ell_1^N$
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In general, the magnitude function of the cuboid
\(A = [0, L_1] \times \cdots \times [0, L_N] \subset \ell^N_1\) is

\[
t \mapsto \sum_{i=0}^{N} 2^{-i} \mu_i(A) t^i
\]

where \(\mu_i\) is \(i\)-dimensional intrinsic volume.
We know: the magnitude function of a cuboid $A \subset \mathbb{R}^n$ is $t \mapsto \sum_{i=0}^{2n} t^i \mu_i(A)$.

Lesson: For this particular class of spaces, the magnitude function encodes many important invariants:

- all the intrinsic volumes
- the dimension.

Conjectural principle: The same is true for a much larger class of spaces, including convex subsets of $\mathbb{R}^n$ with the Euclidean metric.
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Lesson: For this particular class of spaces, the magnitude function encodes many important invariants:

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Pause for reflection

We know: the magnitude function of a cuboid $A \subset \ell_1^N$ is

$$t \mapsto \sum_{i=0}^{N} 2^{-i} \mu_i(A) t^i.$$  

Lesson: For this particular class of spaces, the magnitude function encodes many important invariants:

- all the intrinsic volumes
- the dimension.
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Conjectural principle: The same is true for a much larger class of spaces
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**Lesson:** For this particular class of spaces, the magnitude function encodes many important invariants:

- all the intrinsic volumes
- the dimension.

**Conjectural principle:** The same is true for a much larger class of spaces, including convex subsets of $\mathbb{R}^N$ with the Euclidean metric.
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For a metric space $A$, define

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E.g.: for nondegenerate cuboids $A \subset \ell_1^N$, we have $\dim(A) = N$. 

Theorem

Let $A$ be a compact subset of $\mathbb{R}^N$, with Euclidean metric. Then

$$\dim(A) \leq N$$

with equality if $A$ has nonzero Lebesgue measure.
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The Convex Conjecture

Let $A$ be a compact, convex subset of $\mathbb{R}^N$, with Euclidean metric.

Then $|A| = \sum_{i=0}^{N} \frac{1}{i!} \omega_i \mu_i(A)$

where $\omega_i$ is the volume of the unit $i$-ball.

If this is true then $A$ has magnitude function $t \mapsto \sum_{i=0}^{N} \frac{1}{i!} \omega_i \mu_i(A) \cdot t^i$. 

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So, all of the intrinsic volumes of a convex set (as well as the dimension) can be extracted from its magnitude function.
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- We know that the magnitude function of $A$ has growth $N$
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- Numerical computations support the conjecture.
Summary

Magnitude is a canonical invariant of metric spaces. It appears to subsume the most important invariants of integral geometry. A conjecture states this precisely for convex subsets of $\mathbb{R}^N$. We would like someone here to prove it.

Tomorrow: Magnitude contains more than just known invariants.
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