

The categorical origins of entropy

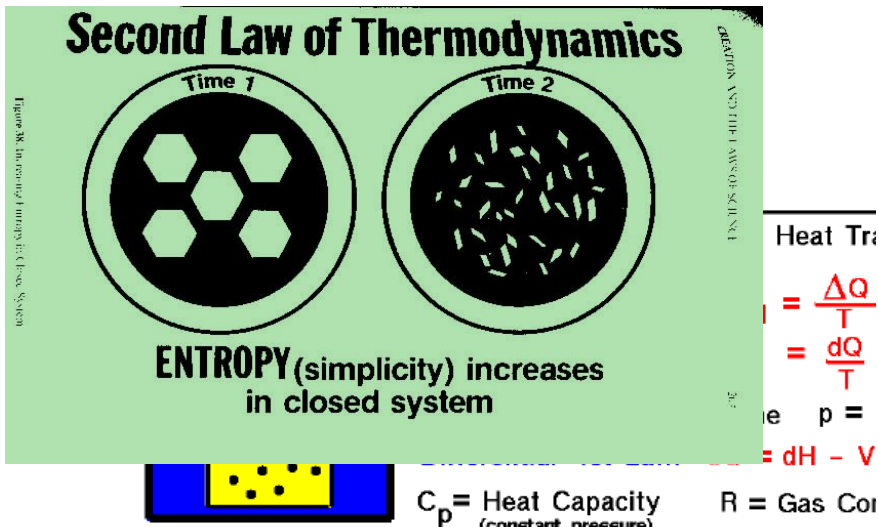
Tom Leinster

University of Edinburgh

These slides: www.maths.ed.ac.uk/~tl/

Entropy in science

Entropy of one kind or another is important in very many branches of science:



Entropy in science

Entropy of one kind or another is important in very many branches of science:

Entropy Approach to the Investigation of Information Capabilities of Adaptive Radio Engineering System in Conditions of Intrasystem Uncertainty

V. V. Skachkov,^{*} V. V. Chepyki,^{**} H. D. Bratchenko,^{***} and A. N. Efymchykov

Odessa State Academy of Technical Regulation and Quality, Odessa, Ukraine



Heat Tra

$$= \frac{\Delta Q}{T}$$

Minimum entropy control of nonlinear ARMA systems over a communication network

Jianhua Zhang · Hong Wang

C_p - Heat Capacity
(constant pressure)

n - Gas Co

Entropy in science

Entropy of one kind or another is important in very many branches of science:

NOISE AND INFORMATION ENTROPY

Entropy, the Central Limit Theorem and the Algebra of the Canonical Commutation Relation

DÉNES PETZ

Mathematical Institute HAS, H-1364 Budapest, PF 127, Hungary

- LOW ENTROPY
- HIGH DEGREE
OF CERTAINTY
- LOW DEGREE
OF UNCERTAINTY

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Entropy in science

Entropy of one kind or another is important in very many branches of science:

BAYESIAN INFERENCE AND MAXIMUM ENTROPY METHODS IN SCIENCE AND ENGINEERING

Proceedings of the 30th International Workshop
on Bayesian Inference and Maximum Entropy
Methods in Science and Engineering

4 - 9 July 2010 Chamonix, France

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Entropy, the
Algebra of

DÉNES PETZ
Mathematical Institute

OVER

Jian

Entropy in science

Entropy of one kind or another is important in very many branches of science:



Entropy in science

Entropy of one kind or another is important in very many branches of science:

Maximum Entropy and Ecology

*A Theory of Abundance, Distribution,
and Energetics*

Impact of a Change of Support on the Assessment of Biodiversity with Shannon Entropy

Didier Josselin¹, Ilene Mahfoud¹, Bruno Fady²



Entropy in science

Entropy of one kind or another is important in very many branches of science:

Roman F. Nalewajski

Reduced communication channels of molecular fragments and their entropy/information bond indices

An entropic characterization of protein interaction networks and cellular robustness

Thomas Manke, Lloyd Demetrius, Martin Vingron

M

Resilience and entropy as indices of robustness of water distribution networks

R. Greco, A. Di Nardo and G. Santonastaso

Entropy in science

Entropy of one kind or another is important in very many branches of science:

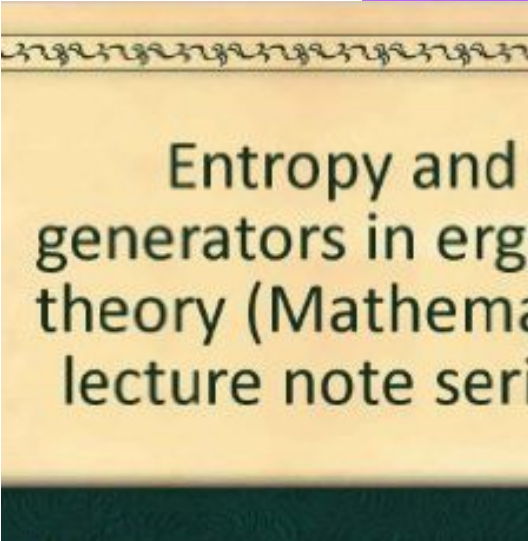
7 Algorithmic entropy and Kolmogorov complexity

Entropy and Quantum Kolmogorov Complexity: A Quantum Brudno's Theorem

Fabio Benatti¹, Tyll Krüger^{2,3}, Markus Müller², Rainer Siegmund-Schultze²,
Arleta Szkoła²

Entropy in science

Entropy of one kind or another is important in very many branches of science:



Entropy and
generators in ergodic
theory (Mathematical
lecture note series)

new mathematical monographs: 18

Entropy in Dynamical
Systems

Tomasz Downarowicz

CAMBRIDGE

Entropy in science

Entropy of one kind or another is important in very many branches of science:

Graham Everest and Paul Vojta
HEIGHTS OF POLYNOMIALS AND ENTROPY IN ALGEBRAIC DYNAMICS

COMBINATORIAL DYNAMICS AND ENTROPY IN DIMENSION ONE
Second Edition
Lindenstrauss and Rokhlin

new mathematical monographs: 18

Entropy in Dynamical Systems
Downarowicz

Investigation of chaotic advection in the atmosphere: the use of topological entropy
Tímea Haszpra

CAMBRIDGE

Entropy in category theory, algebra and topology



The point of this talk

Entropy is notable by its absence from category theory, algebra and topology.

However, we will see that entropy is *inevitable* in pure mathematics: it is there whether we like it or not.

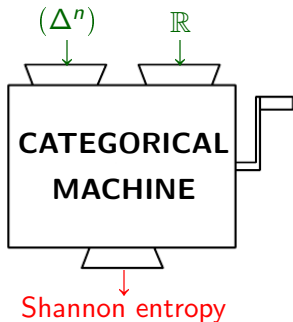


Image: J. Kock

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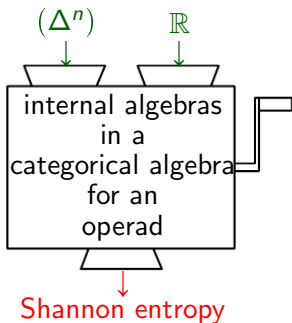


Image: J. Kock

Plan

1. The definition of entropy
2. Operads and their algebras
 3. Internal algebras
 4. The theorem
5. Low-tech corollary

1. The definition of entropy

The definition of entropy

Let $\mathbf{p} = (p_1, \dots, p_n)$ be a probability distribution on $\{1, \dots, n\}$.

That is, let $\mathbf{p} \in [0, 1]^n$ with $\sum_i p_i = 1$.

The **Shannon entropy** of \mathbf{p} is

$$H(\mathbf{p}) = - \sum_{i=1}^n p_i \log p_i$$

(where $0 \log 0 = 0$).

It measures disorder, or information, or expected surprise, or **uniformity**,

For a fixed n :

- the maximum entropy is $H(1/n, 1/n, \dots, 1/n) = \log n$
- the minimum entropy is $H(0, \dots, 0, 1, 0, \dots, 0) = 0$.

Changing the base of the logarithm scales H by a constant factor.

2. Operads and their algebras

The definition of operad

A **[symmetric] operad** \mathcal{O} is a sequence $(O_n)_{n \in \mathbb{N}}$ of sets together with:

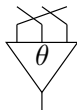
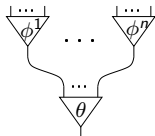
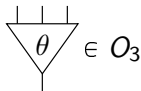
(i) **composition**: for each $k, n_1, \dots, n_k \in \mathbb{N}$, a function

$$\begin{aligned} O_n \times O_{k_1} \times \dots \times O_{k_n} &\longrightarrow O_{k_1 + \dots + k_n} \\ (\theta, \phi^1, \dots, \phi^n) &\longmapsto \theta \circ (\phi^1, \dots, \phi^n) \end{aligned}$$

(ii) **unit**: an element $1 \in O_1$

[(iii) **symmetry**: for each n , an action of S_n on O_n]

satisfying monoid-like axioms.



Examples of operads

1. The **terminal operad** $O = \mathbf{1}$ has $O_n = \{\cdot\}$ for all n .
2. Given monoid M , get operad $O(M)$ with $(O(M))_n = \begin{cases} M & \text{if } n = 1, \\ \emptyset & \text{otherwise.} \end{cases}$
3. The **operad of simplices** Δ , with




$$\Delta_n = \{\text{probability distributions on } \{1, \dots, n\}\} = \Delta^{n-1}.$$

Composition: given

$$\mathbf{p} = (p_1, \dots, p_n), \quad \mathbf{q}^1 = (q_1^1, \dots, q_{k_1}^1), \dots, \mathbf{q}^n = (q_1^n, \dots, q_{k_n}^n),$$

define

$$\mathbf{p} \circ (\mathbf{q}^1, \dots, \mathbf{q}^n) = (p_1 q_1^1, \dots, p_1 q_{k_1}^1, \dots, p_n q_1^n, \dots, p_n q_{k_n}^n).$$

E.g.: $\mathbf{p} =$  , $\mathbf{q}^1 =$  , $\mathbf{q}^2 =$  .

Then $\mathbf{p} \circ (\mathbf{q}^1, \mathbf{q}^2) = (\underbrace{1/12, \dots, 1/12}_6, \underbrace{1/104, \dots, 1/104}_{52}) \in \Delta_{58}$.

Algebras for an operad

Fix an operad O .

An O -algebra is a set A together with a map

$$\bar{\theta}: A^n \longrightarrow A$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying action-like axioms:

- (i) composition, (ii) unit, [(iii) symmetry.]

Examples:

- An algebra for the [symmetric] operad 1 is a [commutative] monoid.
- An $O(M)$ -algebra is an M -set.
- Let $A \subseteq \mathbb{R}^d$ be a convex set. Then A becomes a Δ -algebra as follows: given $\mathbf{p} \in \Delta_n$, define

$$\begin{aligned} \bar{\mathbf{p}}: \quad A^n &\longrightarrow A \\ (\mathbf{a}^1, \dots, \mathbf{a}^n) &\longmapsto \sum_i p_i \mathbf{a}^i. \end{aligned}$$

Categorical algebras for an operad

Fix an operad O .

Extending the definition in the obvious way, we can consider O -algebras in any category \mathcal{A} with finite products (not just **Set**).

A **categorical O -algebra** is an O -algebra in **Cat**.

Explicitly, it is a category \mathbf{A} together with a functor

$$\bar{\theta}: \mathbf{A}^n \longrightarrow \mathbf{A}$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying action-like axioms.

Examples:

- A categorical algebra for the nonsymmetric operad 1 is a strict monoidal category.
- A categorical $O(M)$ -algebra is a category with an M -action.
- Let A be a convex submonoid of $(\mathbb{R}^d, +, 0)$. Viewing A as a one-object category, it is a categorical Δ -algebra in the way defined above.

Maps between categorical algebras for an operad

Fix an operad O and categorical O -algebras \mathbf{B} and \mathbf{A} .

A **lax map** $\mathbf{B} \rightarrow \mathbf{A}$ is a functor $G: \mathbf{B} \rightarrow \mathbf{A}$ together with a natural transformation

$$\begin{array}{ccc} \mathbf{B}^n & \xrightarrow{G^n} & \mathbf{A}^n \\ \bar{\theta} \downarrow & \swarrow \gamma_\theta & \downarrow \bar{\theta} \\ \mathbf{B} & \xrightarrow{G} & \mathbf{A} \end{array}$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying axioms.

Explicitly: it's G together with a map

$$\gamma_{\theta, b^1, \dots, b^n}: \bar{\theta}(Gb^1, \dots, Gb^n) \rightarrow G(\bar{\theta}(b^1, \dots, b^n))$$

for each $\theta \in O_n$ and $b^1, \dots, b^n \in \mathbf{B}$, satisfying naturality and axioms on:

- (i) composition, (ii) unit, [(iii) symmetry.]

3. Internal algebras

Internal algebras in a categorical algebra for an operad

Fix an operad O and a categorical O -algebra \mathbf{A} .

Write $\mathbf{1}$ for the terminal categorical O -algebra.

Definition (Batatin): An **internal algebra** in \mathbf{A} is a lax map $\mathbf{1} \rightarrow \mathbf{A}$.

Explicitly: it's an object $a \in \mathbf{A}$ together with a map

$$\gamma_\theta: \bar{\theta}(a, \dots, a) \longrightarrow a$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying axioms on

- (i) composition, (ii) unit, [(iii) symmetry.]

Examples:

- a. Let $O = 1$ (nonsymmetric). Let \mathbf{A} be a strict monoidal category. An internal O -algebra in \mathbf{A} is just a monoid in \mathbf{A} .
- b. Let $O = O(M)$. Let \mathbf{A} be a category with an M -action. An internal O -algebra in \mathbf{A} is an object $a \in \mathbf{A}$ with a map $\gamma_m: m \cdot a \rightarrow a$ for each $m \in M$, satisfying action-like axioms.

Internal algebras in a categorical algebra for an operad

We fixed an operad O and a categorical O -algebra \mathbf{A} .

Consider the case where \mathbf{A} has only one object, i.e. is a monoid A .

An internal algebra in A then consists of a function

$$\gamma: O_n \longrightarrow A$$

for each $n \in \mathbb{N}$, satisfying axioms on

- (i) composition, (ii) unit, [(iii) symmetry.]

Topologizing everything

Everything so far can be done internally to a category \mathcal{E} with finite limits (instead of **Set**).

So then, $O_n \in \mathcal{E}$, **A** is an internal category in \mathcal{E} , etc.

We take $\mathcal{E} = \mathbf{Top}$.

Explicitly, this means that throughout, we add a condition

(iv) continuity

to the conditions (i)—(iii) that appear repeatedly.

4. *The theorem*

The theorem

Recall: we have

- the (symmetric, topological) operad $\Delta = (\Delta_n)_{n \in \mathbb{N}}$ of simplices
- the (symmetric, topological) categorical Δ -algebra $\mathbb{R} = (\mathbb{R}, +, 0)$.

We just saw that an internal algebra in the categorical Δ -algebra \mathbb{R} consists of functions $\Delta_n \rightarrow \mathbb{R}$ ($n \in \mathbb{N}$) satisfying certain axioms.

Theorem

The internal algebras in the categorical Δ -algebra \mathbb{R} are precisely the scalar multiples of Shannon entropy.

The theorem

Theorem

The internal algebras in the categorical Δ -algebra \mathbb{R} are precisely the scalar multiples of Shannon entropy.

Explicitly, this says: take a sequence of functions $\gamma: \Delta_n \rightarrow \mathbb{R}$ ($n \in \mathbb{N}$).

Then $\gamma = cH$ for some $c \in \mathbb{R}$ if and only if γ satisfies:

- (i) **composition:** $\gamma(\mathbf{p} \circ (\mathbf{q}^1, \dots, \mathbf{q}^n)) = \gamma(\mathbf{p}) + \sum_i p_i \gamma(\mathbf{q}^i)$
- (ii) **unit:** $\gamma((1)) = 0$
- (iii) **symmetry:** $\gamma((p_1, \dots, p_n)) = \gamma((p_{\sigma(1)}, \dots, p_{\sigma(n)}))$ ($\sigma \in S_n$)
- (iv) **continuity:** each function γ is continuous.

Proof: This explicit form is equivalent to a 1956 theorem of Faddeev. □

5. Low-tech corollary

(with John Baez and Tobias Fritz)

The free categorical algebra containing an internal algebra

Thought: A monoid in a monoidal category \mathbf{A} is the same thing as a lax monoidal functor $\mathbf{1} \longrightarrow \mathbf{A}$.

But it's also the same as a strict monoidal functor $\mathbf{D} \longrightarrow \mathbf{A}$, where $\mathbf{D} = (\text{finite ordinals})$ is the free monoidal category containing a monoid.

We can try to imitate this for algebras for other operads, such as Δ .

Fact: The free categorical Δ -algebra containing an internal algebra is (nearly) the category **FinProb** in which:

- an object (X, \mathbf{p}) is a finite set X with a probability measure \mathbf{p}
- the maps are the measure-preserving maps ('deterministic processes').

Thus, an internal Δ -algebra in \mathbb{R} is a functor $\mathbf{FinProb} \longrightarrow (\mathbb{R}, +, 0)$ satisfying certain axioms.

An explicit characterization of entropy

Corollary (with John Baez and Tobias Fritz)

Let $L: \{\text{maps in } \mathbf{FinProb}\} \rightarrow \mathbb{R}$ be a function that 'measures information loss', that is, satisfies:

- $L(g \circ f) = L(f) + L(g)$
- $L(\lambda f \oplus (1 - \lambda)f') = \lambda L(f) + (1 - \lambda)L(f')$
- $L(f) = 0$ if f is invertible
- L is continuous.

$$\begin{array}{c} \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \\ \cdot \xrightarrow[\text{f'}]{f} \cdot \end{array}$$

Then there is some $c \in \mathbb{R}$ such that

$$L\left((X, \mathbf{p}) \xrightarrow{f} (Y, \mathbf{q})\right) = c \cdot (H(\mathbf{p}) - H(\mathbf{q}))$$

for all f .

Summary

Summary

- Given an operad O and an O -algebra \mathbf{A} in \mathbf{Cat} , there is a general concept of **internal algebra** in \mathbf{A} .
 - Applied to the terminal operad 1 , this gives the concept of (internal) **monoid** in a monoidal category.
 - Applied to the operad Δ of simplices and its algebra $(\mathbb{R}, +, 0)$ in \mathbf{Cat} , it gives the concept of **Shannon entropy**.

In short: entropy is inevitable.

- Given an operad O , we can form the **free categorical O -algebra containing an internal algebra**.
 - When $O = 1$, this is the category of **finite ordinals**.
 - When $O = \Delta$, this is the category of **finite probability spaces** (nearly). That observation leads to a new and entirely explicit characterization of Shannon entropy.