The categorical origins of entropy

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These slides: www.maths.ed.ac.uk/~tl/
Entropy in science

Entropy of one kind or another is important in very many branches of science:

**Second Law of Thermodynamics**

- ENTROPY (simplicity) increases in closed system

Heat Transfer:

\[ Q = \frac{\Delta T}{T} \]

Differential for Each State:

\[ dQ = dH - dV \]

\[ C_p = \text{Heat Capacity} \]

\[ R = \text{Gas Constant} \]
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Entropy of one kind or another is important in very many branches of science:

- Reduced communication channels of molecular fragments and their entropy/information bond indices
  
  Roman F. Nalewajski

- An entropic characterization of protein interaction networks and cellular robustness
  
  Thomas Manke, Lloyd Demetrius, Martin Vingron

- Resilience and entropy as indices of robustness of water distribution networks
  
  R. Greco, A. Di Nardo and G. Santonastaso
Entropy in science

Entropy of one kind or another is important in very many branches of science:

Algorithmic entropy and Kolmogorov complexity

Entropy and Quantum Kolmogorov Complexity: A Quantum Brudno’s Theorem

Fabio Benatti\textsuperscript{1}, Tyll Krüger\textsuperscript{2,3}, Markus Müller\textsuperscript{2}, Rainer Siegmund-Schultze\textsuperscript{2}, Arleta Szołła\textsuperscript{2}
Entropy in science

Entropy of one kind or another is important in very many branches of science:
Entropy in science

Entropy of one kind or another is important in very many branches of science:
Entropy in category theory, algebra and topology
The point of this talk

Entropy is notable by its absence from category theory, algebra and topology. However, we will see that entropy is \textit{inevitable} in pure mathematics: it is there whether we like it or not.

![Diagram of a categorical machine with Shannon entropy](Image: J. Kock)
The point of this talk

Entropy is notable by its absence from category theory, algebra and topology. However, we will see that entropy is *inevitable* in pure mathematics: it is there whether we like it or not.

![Diagram](Image: J. Kock)
Plan

1. The definition of entropy
2. Operads and their algebras
3. Internal algebras
4. The theorem
5. Low-tech corollary
1. The definition of entropy
The definition of entropy

Let \( \mathbf{p} = (p_1, \ldots, p_n) \) be a probability distribution on \( \{1, \ldots, n\} \). That is, let \( \mathbf{p} \in [0, 1]^n \) with \( \sum_i p_i = 1 \).

The Shannon entropy of \( \mathbf{p} \) is

\[
H(\mathbf{p}) = - \sum_{i=1}^{n} p_i \log p_i
\]

(where \( 0 \log 0 = 0 \)).

It measures disorder, or information, or expected surprise, or uniformity, \ldots

For a fixed \( n \):

- the maximum entropy is \( H(1/n, 1/n, \ldots, 1/n) = \log n \)
- the minimum entropy is \( H(0, \ldots, 0, 1, 0, \ldots, 0) = 0 \).

Changing the base of the logarithm scales \( H \) by a constant factor.
2. Operads and their algebras
The definition of operad

A [symmetric] operad $O$ is a sequence $(O_n)_{n \in \mathbb{N}}$ of sets together with:

(i) **composition**: for each $k, n_1, \ldots, n_k \in \mathbb{N}$, a function

\[
O_n \times O_{k_1} \times \cdots \times O_{k_n} \rightarrow O_{k_1 + \cdots + k_n}
\]

\[
(\theta, \phi^1, \ldots, \phi^n) \rightarrow \theta \circ (\phi^1, \ldots, \phi^n)
\]

(ii) **unit**: an element $1 \in O_1$

[(iii) **symmetry**: for each $n$, an action of $S_n$ on $O_n]

satisfying monoid-like axioms.
Examples of operads

1. The terminal operad $O = 1$ has $O_n = \{\cdot\}$ for all $n$.

2. Given monoid $M$, get operad $O(M)$ with $(O(M))_n = \begin{cases} M & \text{if } n = 1, \\ \emptyset & \text{otherwise.} \end{cases}$

3. The operad of simplices $\Delta$, with

   \[
   \Delta_n = \{\text{probability distributions on } \{1, \ldots, n\}\} = \Delta^{n-1}.
   \]

Composition: given

\[
p = (p_1, \ldots, p_n), \quad q^1 = (q_1^1, \ldots, q_{k_1}^1), \ldots, \quad q^n = (q_1^n, \ldots, q_{k_n}^n),
\]

define

\[
p \circ (q^1, \ldots, q^n) = (p_1 q_1^1, \ldots, p_1 q_{k_1}^1, \ldots, p_n q_1^n, \ldots, p_n q_{k_n}^n).
\]

E.g.: \[
p = \left( \begin{array}{c} \text{coin} \\ \text{coin} \end{array} \right), \quad q^1 = \text{dice}, \quad q^2 = \clubsuit.
\]

Then \[
p \circ (q^1, q^2) = (1/12, \ldots, 1/12, 1/104, \ldots, 1/104) \in \Delta_{58}.
\]
Algebras for an operad

Fix an operad $O$.

An $O$-algebra is a set $A$ together with a map

$$\overline{\theta} : A^n \rightarrow A$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying action-like axioms:

(i) composition, (ii) unit, [(iii) symmetry.]

Examples:

a. An algebra for the [symmetric] operad 1 is a [commutative] monoid.

b. An $O(M)$-algebra is an $M$-set.

c. Let $A \subseteq \mathbb{R}^d$ be a convex set. Then $A$ becomes a $\Delta$-algebra as follows:

given $p \in \Delta_n$, define

$$\overline{p} : A^n \rightarrow A$$

$$(a^1, \ldots, a^n) \mapsto \sum_i p_i a^i.$$
Fix an operad $O$.

Extending the definition in the obvious way, we can consider $O$-algebras in any category $\mathcal{A}$ with finite products (not just $\text{Set}$).

A categorical $O$-algebra is an $O$-algebra in $\text{Cat}$.

Explicitly, it is a category $\mathcal{A}$ together with a functor

$$\bar{\theta} : \mathcal{A}^n \rightarrow \mathcal{A}$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying action-like axioms.

Examples:

a. A categorical algebra for the nonsymmetric operad 1 is a strict monoidal category.

b. A categorical $O(M)$-algebra is a category with an $M$-action.

c. Let $A$ be a convex submonoid of $(\mathbb{R}^d, +, 0)$. Viewing $A$ as a one-object category, it is a categorical $\Delta$-algebra in the way defined above.
Maps between categorical algebras for an operad

Fix an operad $O$ and categorical $O$-algebras $B$ and $A$.

A lax map $B \to A$ is a functor $G: B \to A$ together with a natural transformation

\[
\begin{array}{ccc}
B^n & \xrightarrow{G^n} & A^n \\
\downarrow \theta & & \downarrow \theta \\
\text{B} & \xrightarrow{G} & \text{A}
\end{array}
\]

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying axioms.

Explicitly: it’s $G$ together with a map

\[
\gamma_{\theta,b^1,\ldots,b^n}: \theta (Gb^1, \ldots, Gb^n) \to G(\theta(b^1, \ldots, b^n))
\]

for each $\theta \in O_n$ and $b^1, \ldots, b^n \in B$, satisfying naturality and axioms on:

(i) composition, (ii) unit, [(iii) symmetry.]
3. *Internal algebras*
Internal algebras in a categorical algebra for an operad

Fix an operad $O$ and a categorical $O$-algebra $A$.

Write $\mathbf{1}$ for the terminal categorical $O$-algebra.

Definition (Batanin): An internal algebra in $A$ is a lax map $\mathbf{1} \rightarrow A$.

Explicitly: it’s an object $a \in A$ together with a map

$$\gamma_\theta : \bar{\theta}(a, \ldots, a) \rightarrow a$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying axioms on

(i) composition,  
(ii) unit,  
[(iii) symmetry.]

Examples:

a. Let $O = 1$ (nonsymmetric). Let $A$ be a strict monoidal category.
   An internal $O$-algebra in $A$ is just a monoid in $A$.

b. Let $O = O(M)$. Let $A$ be a category with an $M$-action.
   An internal $O$-algebra in $A$ is an object $a \in A$ with a map
   $\gamma_m : m \cdot a \rightarrow a$ for each $m \in M$, satisfying action-like axioms.
We fixed an operad $O$ and a categorical $O$-algebra $A$. Consider the case where $A$ has only one object, i.e. is a monoid $A$. An internal algebra in $A$ then consists of a function

$$\gamma: O_n \rightarrow A$$

for each $n \in \mathbb{N}$, satisfying axioms on

(i) composition,     (ii) unit,     [(iii) symmetry.]
Topologizing everything

Everything so far can be done internally to a category $\mathcal{E}$ with finite limits (instead of $\text{Set}$).

So then, $O_n \in \mathcal{E}$, $\mathcal{A}$ is an internal category in $\mathcal{E}$, etc.

We take $\mathcal{E} = \text{Top}$.

Explicitly, this means that throughout, we add a condition

(iv) continuity

to the conditions (i)—(iii) that appear repeatedly.
4. The theorem
Theorem

Recall: we have

- the (symmetric, topological) operad $\Delta = (\Delta_n)_{n \in \mathbb{N}}$ of simplices
- the (symmetric, topological) categorical $\Delta$-algebra $\mathbb{R} = (\mathbb{R}, +, 0)$.

We just saw that an internal algebra in the categorical $\Delta$-algebra $\mathbb{R}$ consists of functions $\Delta_n \rightarrow \mathbb{R}$ ($n \in \mathbb{N}$) satisfying certain axioms.

**Theorem**

The internal algebras in the categorical $\Delta$-algebra $\mathbb{R}$ are precisely the scalar multiples of Shannon entropy.
The internal algebras in the categorical $\Delta$-algebra $\mathbb{R}$ are precisely the scalar multiples of Shannon entropy.

Explicitly, this says: take a sequence of functions $\gamma: \Delta_n \to \mathbb{R}$ ($n \in \mathbb{N}$). Then $\gamma = cH$ for some $c \in \mathbb{R}$ if and only if $\gamma$ satisfies:

(i) composition: $\gamma(p \circ (q^1, \ldots, q^n)) = \gamma(p) + \sum_i p_i \gamma(q^i)$

(ii) unit: $\gamma((1)) = 0$

(iii) symmetry: $\gamma((p_1, \ldots, p_n)) = \gamma((p_{\sigma(1)}, \ldots, p_{\sigma(n)}))$ ($\sigma \in S_n$)

(iv) continuity: each function $\gamma$ is continuous.

Proof: This explicit form is equivalent to a 1956 theorem of Faddeev.
5. Low-tech corollary
(with John Baez and Tobias Fritz)
The free categorical algebra containing an internal algebra

**Thought:** A monoid in a monoidal category $A$ is the same thing as a lax monoidal functor $1 \to A$.

But it’s also the same as a strict monoidal functor $D \to A$, where $D = (\text{finite ordinals})$ is the free monoidal category containing a monoid.

We can try to imitate this for algebras for other operads, such as $\Delta$.

**Fact:** The free categorical $\Delta$-algebra containing an internal algebra is (nearly) the category $\text{FinProb}$ in which:

- an object $(X, p)$ is a finite set $X$ with a probability measure $p$
- the maps are the measure-preserving maps (‘deterministic processes’).

Thus, an internal $\Delta$-algebra in $\mathbb{R}$ is a functor $\text{FinProb} \to (\mathbb{R}, +, 0)$ satisfying certain axioms.
An explicit characterization of entropy

Corollary (with John Baez and Tobias Fritz)

Let $L: \{\text{maps in FinProb}\} \rightarrow \mathbb{R}$ be a function that ‘measures information loss’, that is, satisfies:

- $L(g \circ f) = L(f) + L(g)$
- $L(\lambda f \oplus (1 - \lambda) f') = \lambda L(f) + (1 - \lambda) L(f')$
- $L(f) = 0$ if $f$ is invertible
- $L$ is continuous.

Then there is some $c \in \mathbb{R}$ such that

$$L\left( (X, p) \xrightarrow{f} (Y, q) \right) = c \cdot (H(p) - H(q))$$

for all $f$. 
Summary
Summary

- Given an operad $O$ and an $O$-algebra $A$ in $\textbf{Cat}$, there is a general concept of internal algebra in $A$.
  - Applied to the terminal operad 1, this gives the concept of (internal) monoid in a monoidal category.
  - Applied to the operad $\Delta$ of simplices and its algebra $(\mathbb{R}, +, 0)$ in $\textbf{Cat}$, it gives the concept of Shannon entropy.

In short: entropy is inevitable.

- Given an operad $O$, we can form the free categorical $O$-algebra containing an internal algebra.
  - When $O = 1$, this is the category of finite ordinals.
  - When $O = \Delta$, this is the category of finite probability spaces (nearly). That observation leads to a new and entirely explicit characterization of Shannon entropy.