# The categorical origins of entropy

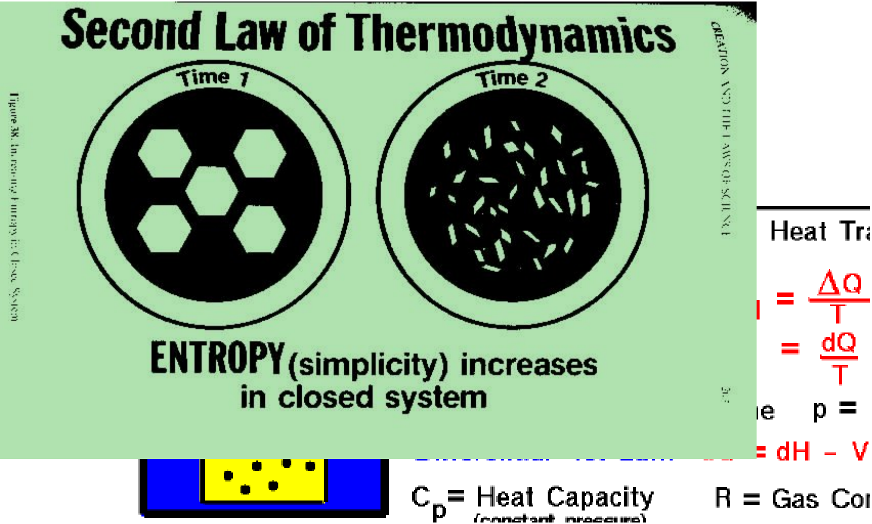
Tom Leinster

University of Edinburgh

These slides: www.maths.ed.ac.uk/$\sim$tl/

# Entropy in science

Entropy of one kind or another is important in very many branches of science:



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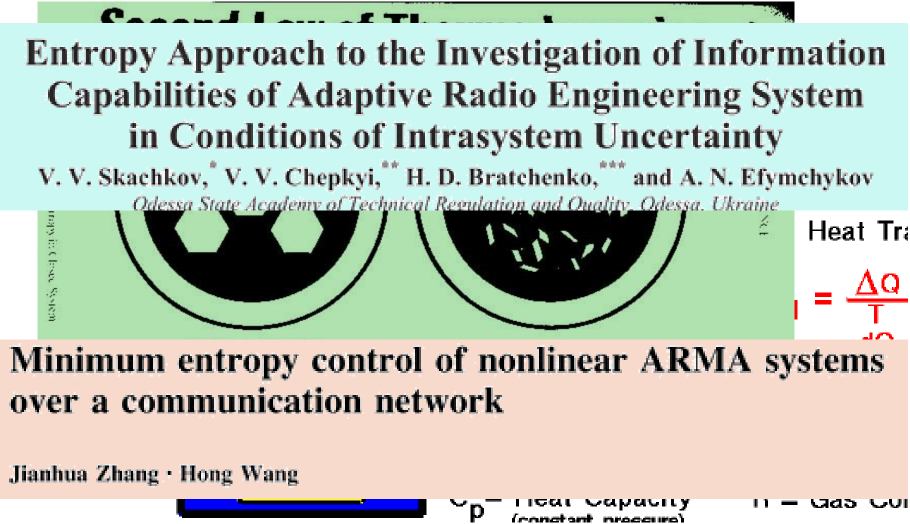
**Entropy Approach to the Investigation of Information Capabilities of Adaptive Radio Engineering System in Conditions of Intrasystem Uncertainty**

V. V. Skachkov,* V. V. Chepkyi,** H. D. Bratchenko,*** and A. N. Efymchykov

*Odessa State Academy of Technical Regulation and Quality, Odessa, Ukraine*

**Minimum entropy control of nonlinear ARMA systems over a communication network**

**Jianhua Zhang · Hong Wang**

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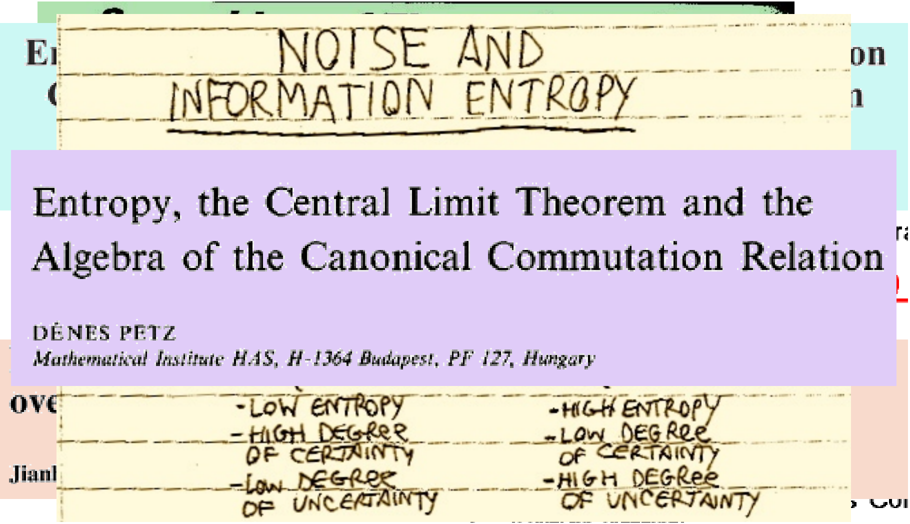

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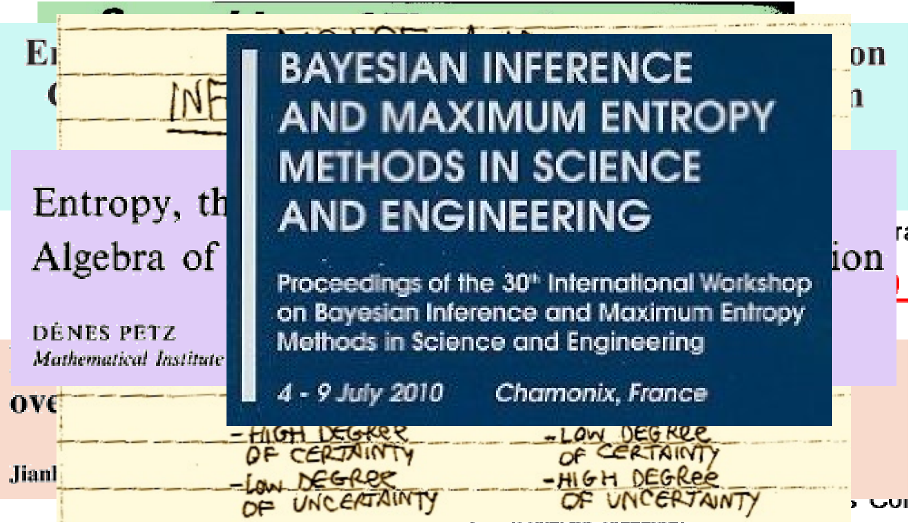

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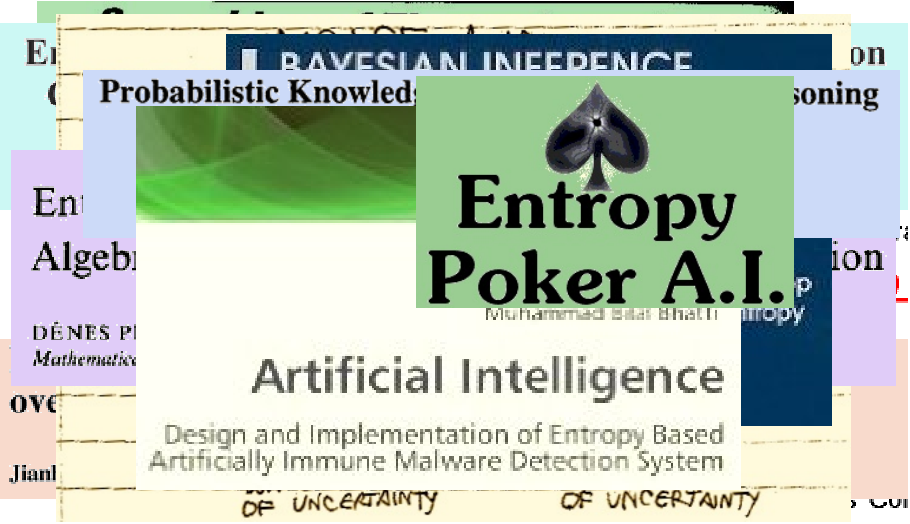

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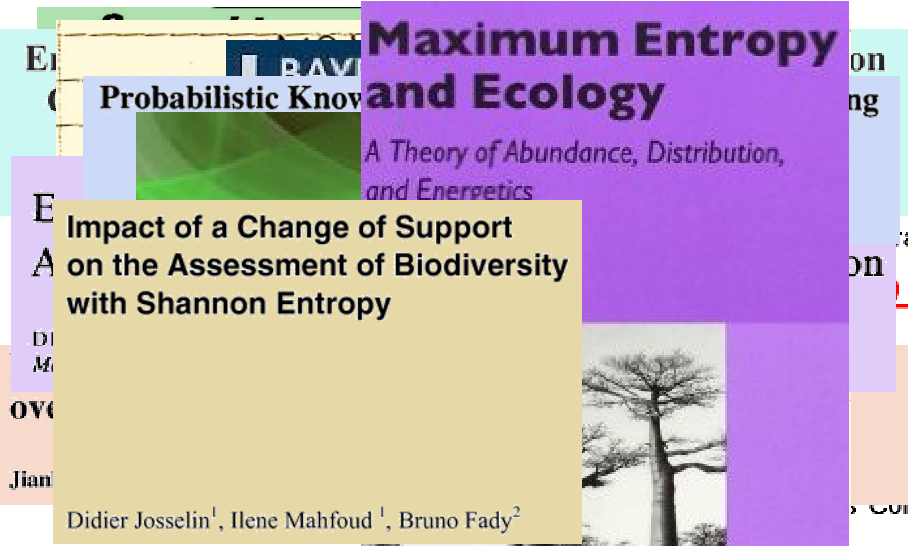

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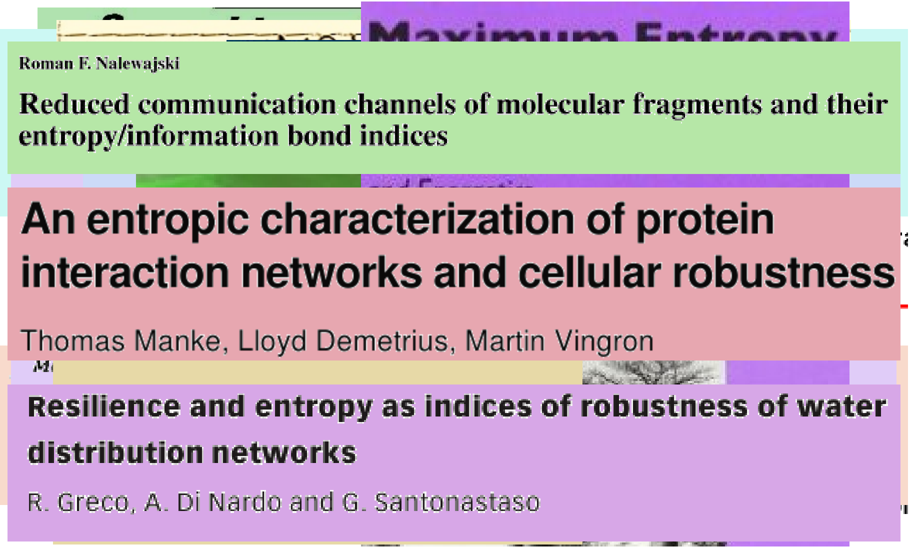

Roman F. Nalewajski

**Reduced communication channels of molecular fragments and their entropy/information bond indices**

**An entropic characterization of protein interaction networks and cellular robustness**

Thomas Manke, Lloyd Demetrius, Martin Vingron

**Resilience and entropy as indices of robustness of water distribution networks**

R. Greco, A. Di Nardo and G. Santonastaso

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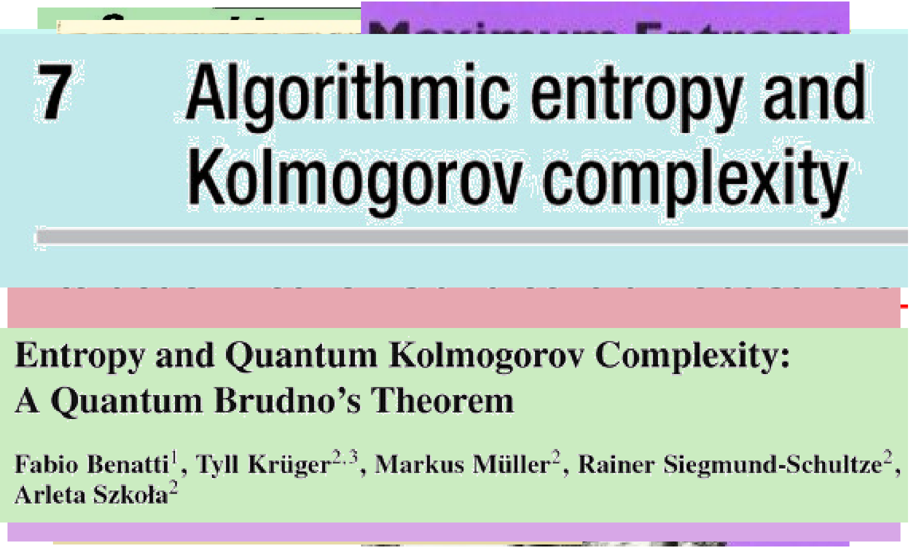

7 Algorithmic entropy and Kolmogorov complexity

Entropy and Quantum Kolmogorov Complexity: A Quantum Brudno's Theorem

Fabio Benatti[1], Tyll Krüger[2,3], Markus Müller[2], Rainer Siegmund-Schultze[2], Arleta Szkoła[2]

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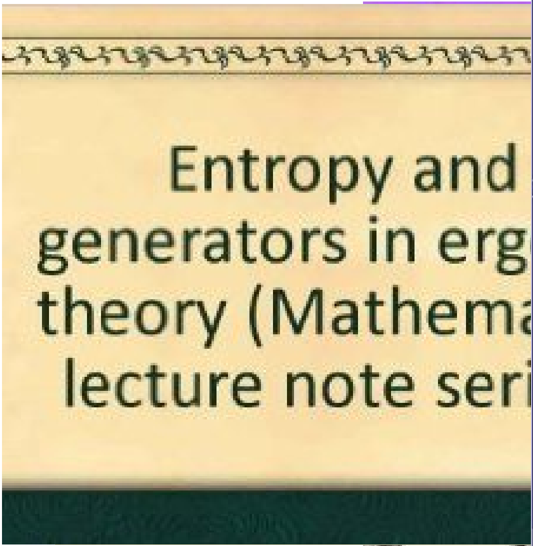


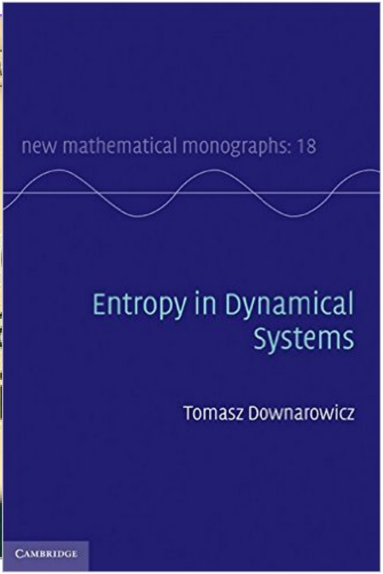

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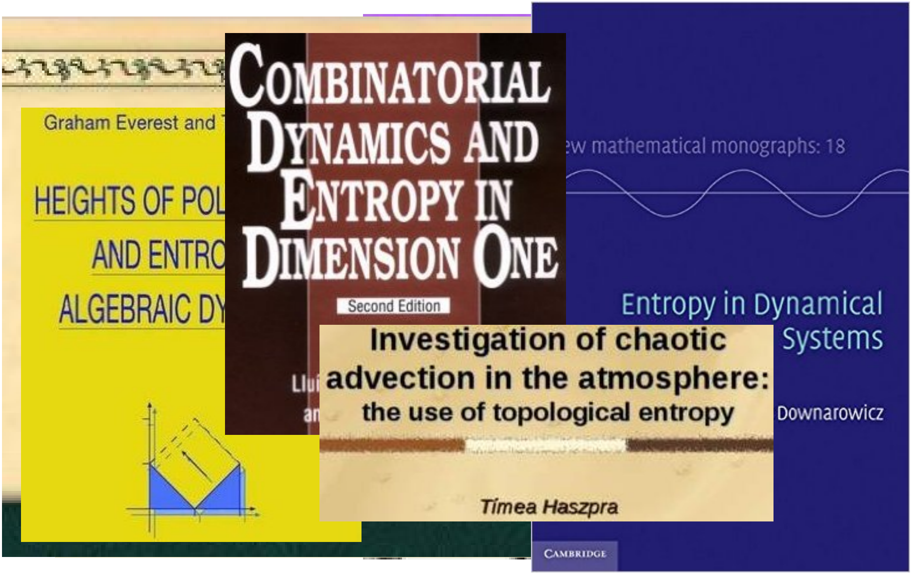

# Entropy in category theory, algebra and topology

?

# The point of this talk

Entropy is notable by its absence from category theory, algebra and topology.

However, we will see that entropy is *inevitable* in pure mathematics:
it is there whether we like it or not.

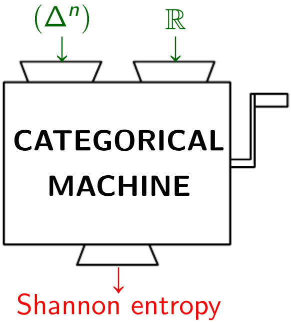


Image: J. Kock

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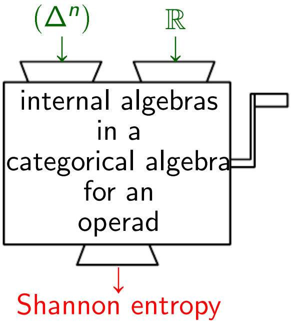



Image: J. Kock

# Plan

1. The definition of entropy
2. Operads and their algebras
3. Internal algebras
4. The theorem
5. Low-tech corollary

# 1. The definition of entropy

# The definition of entropy

Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a probability distribution on $\{1, \ldots, n\}$.
That is, let $\mathbf{p} \in [0,1]^n$ with $\sum_i p_i = 1$.

The Shannon entropy of $\mathbf{p}$ is

$$H(\mathbf{p}) = -\sum_{i=1}^{n} p_i \log p_i$$

(where $0 \log 0 = 0$).

It measures disorder, or information, or expected surprise, or uniformity, ....

For a fixed $n$:

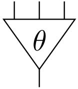
- the maximum entropy is $H(1/n, 1/n, \ldots, 1/n) = \log n$
- the minimum entropy is $H(0, \ldots, 0, 1, 0, \ldots, 0) = 0$.

Changing the base of the logarithm scales $H$ by a constant factor.

# 2. Operads and their algebras

# The definition of operad

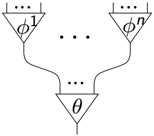
A [symmetric] operad $O$ is a sequence $(O_n)_{n \in \mathbb{N}}$ of sets together with:




$\in O_3$

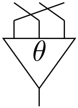
(i) composition: for each $k, n_1, \ldots, n_k \in \mathbb{N}$, a function

$$\begin{array}{ccc} O_n \times O_{k_1} \times \cdots \times O_{k_n} & \longrightarrow & O_{k_1 + \cdots + k_n} \\ (\theta, \phi^1, \ldots, \phi^n) & \longmapsto & \theta \circ (\phi^1, \ldots, \phi^n) \end{array}$$




(ii) unit: an element $1 \in O_1$

[(iii) symmetry: for each $n$, an action of $S_n$ on $O_n$]




satisfying monoid-like axioms.

# Examples of operads

1. The terminal operad $O = 1$ has $O_n = \{\cdot\}$ for all $n$.
2. Given monoid $M$, get operad $O(M)$ with $(O(M))_n = \begin{cases} M & \text{if } n = 1, \\ \varnothing & \text{otherwise.} \end{cases}$
3. The operad of simplices $\Delta$, with

$$\Delta_n = \Big\{\text{probability distributions on } \{1, \ldots, n\}\Big\} = \Delta^{n-1}.$$

Composition: given

$$\mathbf{p} = (p_1, \ldots, p_n), \qquad \mathbf{q}^1 = (q_1^1, \ldots, q_{k_1}^1), \ \ldots, \ \mathbf{q}^n = (q_1^n, \ldots, q_{k_n}^n),$$

define

$$\mathbf{p} \circ (\mathbf{q}^1, \ldots, \mathbf{q}^n) = (p_1 q_1^1, \ldots, p_1 q_{k_1}^1, \ \ldots, \ p_n q_1^n, \ldots, p_n q_{k_n}^n).$$

E.g.: $\mathbf{p} =$  , $\mathbf{q}^1 =$  , $\mathbf{q}^2 =$  .

Then $\mathbf{p} \circ (\mathbf{q}^1, \mathbf{q}^2) = (\underbrace{1/12, \ldots, 1/12}_{6}, \underbrace{1/104, \ldots, 1/104}_{52}) \in \Delta_{58}$.

# Algebras for an operad

Fix an operad $O$.

An $O$-algebra is a set $A$ together with a map

$$\overline{\theta} \colon A^n \longrightarrow A$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying action-like axioms:

(i) composition, (ii) unit, [(iii) symmetry.]

Examples:

a. An algebra for the [symmetric] operad 1 is a [commutative] monoid.
b. An $O(M)$-algebra is an $M$-set.
c. Let $A \subseteq \mathbb{R}^d$ be a convex set. Then $A$ becomes a $\Delta$-algebra as follows: given $\mathbf{p} \in \Delta_n$, define

$$\begin{array}{rccc} \overline{\mathbf{p}} \colon & A^n & \longrightarrow & A \\ & (\mathbf{a}^1, \ldots, \mathbf{a}^n) & \longmapsto & \sum_i p_i \mathbf{a}^i. \end{array}$$

# Categorical algebras for an operad

Fix an operad $O$.

Extending the definition in the obvious way, we can consider $O$-algebras in any category $\mathscr{A}$ with finite products (not just **Set**).

A categorical $O$-algebra is an $O$-algebra in **Cat**.

Explicitly, it is a category $\mathbf{A}$ together with a functor

$$\overline{\theta} \colon \mathbf{A}^n \longrightarrow \mathbf{A}$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying action-like axioms.

Examples:

a. A categorical algebra for the nonsymmetric operad 1 is a strict monoidal category.
b. A categorical $O(M)$-algebra is a category with an $M$-action.
c. Let $A$ be a convex submonoid of $(\mathbb{R}^d, +, 0)$. Viewing $A$ as a one-object category, it is a categorical $\Delta$-algebra in the way defined above.

# Maps between categorical algebras for an operad

Fix an operad $O$ and categorical $O$-algebras $\mathbf{B}$ and $\mathbf{A}$.

A lax map $\mathbf{B} \longrightarrow \mathbf{A}$ is a functor $G \colon \mathbf{B} \longrightarrow \mathbf{A}$ together with a natural transformation

$$\begin{tikzcd} \mathbf{B}^n \arrow[r, "G^n"] \arrow[d, "\overline{\theta}"'] \arrow[dr, phantom, "\swArrow \gamma_\theta"] & \mathbf{A}^n \arrow[d, "\overline{\theta}"] \\ \mathbf{B} \arrow[r, "G"'] & \mathbf{A} \end{tikzcd}$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying axioms.

Explicitly: it's $G$ together with a map

$$\gamma_{\theta,b^1,\ldots,b^n} \colon \overline{\theta}(Gb^1, \ldots, Gb^n) \longrightarrow G(\overline{\theta}(b^1, \ldots, b^n))$$

for each $\theta \in O_n$ and $b^1, \ldots, b^n \in \mathbf{B}$, satisfying naturality and axioms on:

(i) composition, (ii) unit, [(iii) symmetry.]

# 3. Internal algebras

# Internal algebras in a categorical algebra for an operad

Fix an operad $O$ and a categorical $O$-algebra $\mathbf{A}$.

Write $\mathbf{1}$ for the terminal categorical $O$-algebra.

Definition (Batanin): An internal algebra in $\mathbf{A}$ is a lax map $\mathbf{1} \longrightarrow \mathbf{A}$.

Explicitly: it's an object $a \in \mathbf{A}$ together with a map

$$\gamma_\theta : \overline{\theta}(a, \ldots, a) \longrightarrow a$$

for each $n \in \mathbb{N}$ and $\theta \in O_n$, satisfying axioms on

(i) composition, (ii) unit, [(iii) symmetry.]

Examples:

a. Let $O = 1$ (nonsymmetric). Let $\mathbf{A}$ be a strict monoidal category. An internal $O$-algebra in $\mathbf{A}$ is just a monoid in $\mathbf{A}$.
b. Let $O = O(M)$. Let $\mathbf{A}$ be a category with an $M$-action. An internal $O$-algebra in $\mathbf{A}$ is an object $a \in \mathbf{A}$ with a map $\gamma_m : m \cdot a \longrightarrow a$ for each $m \in M$, satisfying action-like axioms.

# Internal algebras in a categorical algebra for an operad

We fixed an operad $O$ and a categorical $O$-algebra $\mathbf{A}$.

Consider the case where $\mathbf{A}$ has only one object, i.e. is a monoid $A$.

An internal algebra in $A$ then consists of a function

$$\gamma \colon O_n \longrightarrow A$$

for each $n \in \mathbb{N}$, satisfying axioms on

(i) composition, (ii) unit, [(iii) symmetry.]

# Topologizing everything

Everything so far can be done internally to a category $\mathscr{E}$ with finite limits (instead of **Set**).

So then, $O_n \in \mathscr{E}$, $\mathbf{A}$ is an internal category in $\mathscr{E}$, etc.

We take $\mathscr{E} = \mathbf{Top}$.

Explicitly, this means that throughout, we add a condition

(iv) continuity

to the conditions (i)—(iii) that appear repeatedly.

# *4. The theorem*

# The theorem

Recall: we have

- the (symmetric, topological) operad $\Delta = (\Delta_n)_{n\in\mathbb{N}}$ of simplices
- the (symmetric, topological) categorical $\Delta$-algebra $\mathbb{R} = (\mathbb{R}, +, 0)$.

We just saw that an internal algebra in the categorical $\Delta$-algebra $\mathbb{R}$ consists of functions $\Delta_n \longrightarrow \mathbb{R} \quad (n \in \mathbb{N})$ satisfying certain axioms.

## Theorem

*The internal algebras in the categorical $\Delta$-algebra $\mathbb{R}$ are precisely the scalar multiples of Shannon entropy.*

# The theorem

## Theorem

*The internal algebras in the categorical $\Delta$-algebra $\mathbb{R}$ are precisely the scalar multiples of Shannon entropy.*

Explicitly, this says: take a sequence of functions $\gamma\colon \Delta_n \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$. Then $\gamma = cH$ for some $c \in \mathbb{R}$ if and only if $\gamma$ satisfies:

(i) composition: $\gamma\big(\mathbf{p} \circ (\mathbf{q}^1, \ldots, \mathbf{q}^n)\big) = \gamma(\mathbf{p}) + \sum_i p_i \gamma(\mathbf{q}^i)$

(ii) unit: $\gamma\big((1)\big) = 0$

(iii) symmetry: $\gamma\big((p_1, \ldots, p_n)\big) = \gamma\big((p_{\sigma(1)}, \ldots, p_{\sigma(n)})\big)$ $(\sigma \in S_n)$

(iv) continuity: each function $\gamma$ is continuous.

Proof: This explicit form is equivalent to a 1956 theorem of Faddeev. $\square$

# 5. Low-tech corollary

*(with John Baez and Tobias Fritz)*

# The free categorical algebra containing an internal algebra

Thought: A monoid in a monoidal category $\mathbf{A}$ is the same thing as a lax monoidal functor $\mathbf{1} \longrightarrow \mathbf{A}$.

But it's also the same as a strict monoidal functor $\mathbf{D} \longrightarrow \mathbf{A}$, where $\mathbf{D} =$ (finite ordinals) is the free monoidal category containing a monoid.

We can try to imitate this for algebras for other operads, such as $\Delta$.

Fact: The free categorical $\Delta$-algebra containing an internal algebra is (nearly) the category **FinProb** in which:

- an object $(X, \mathbf{p})$ is a finite set $X$ with a probability measure $\mathbf{p}$
- the maps are the measure-preserving maps ('deterministic processes').

Thus, an internal $\Delta$-algebra in $\mathbb{R}$ is a functor $\mathbf{FinProb} \longrightarrow (\mathbb{R}, +, 0)$ satisfying certain axioms.

# An explicit characterization of entropy

## Corollary (with John Baez and Tobias Fritz)

*Let* $L\colon \{\textit{maps in } \mathbf{FinProb}\} \to \mathbb{R}$ *be a function that 'measures information loss', that is, satisfies:*

- $L(g \circ f) = L(f) + L(g)$ $\qquad \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$
- $L(\lambda f \oplus (1-\lambda) f') = \lambda L(f) + (1-\lambda) L(f')$ $\qquad : \overset{f}{\underset{f'}{\rightrightarrows}} :$
- $L(f) = 0$ *if* $f$ *is invertible*
- $L$ *is continuous.*

*Then there is some* $c \in \mathbb{R}$ *such that*

$$L\Big( (X, \mathbf{p}) \xrightarrow{f} (Y, \mathbf{q}) \Big) = c \cdot \big( H(\mathbf{p}) - H(\mathbf{q}) \big)$$

*for all* $f$.

# *Summary*

# Summary

- Given an operad $O$ and an $O$-algebra $\mathbf{A}$ in $\mathbf{Cat}$, there is a general concept of internal algebra in $\mathbf{A}$.
  - Applied to the terminal operad $1$, this gives the concept of (internal) monoid in a monoidal category.
  - Applied to the operad $\Delta$ of simplices and its algebra $(\mathbb{R}, +, 0)$ in $\mathbf{Cat}$, it gives the concept of Shannon entropy.

  In short: entropy is inevitable.

- Given an operad $O$, we can form the free categorical $O$-algebra containing an internal algebra.
  - When $O = 1$, this is the category of finite ordinals.
  - When $O = \Delta$, this is the category of finite probability spaces (nearly). That observation leads to a new and entirely explicit characterization of Shannon entropy.