Magnitude

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The idea

For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies

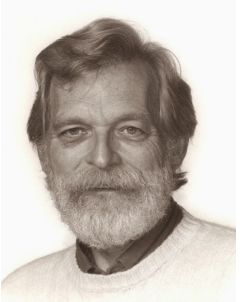
$$|S \cup T| = |S| + |T| - |S \cap T|$$
$$|S \times T| = |S| \times |T|.$$

- Subsets of $\mathbb{R}^n$ have volume. It satisfies

$$\mathrm{vol}(S \cup T) = \mathrm{vol}(S) + \mathrm{vol}(T) - \mathrm{vol}(S \cap T)$$
$$\mathrm{vol}(S \times T) = \mathrm{vol}(S) \times \mathrm{vol}(T).$$

- Topological spaces have Euler characteristic. It satisfies

$$\chi(S \cup T) = \chi(S) + \chi(T) - \chi(S \cap T) \quad \text{(under hypotheses)}$$
$$\chi(S \times T) = \chi(S) \times \chi(T).$$



Stephen Schanuel:
Euler characteristic is the topological analogue of cardinality.

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- Topological spaces have Euler characteristic. It satisfies

$$\chi(S \cup T) = \chi(S) + \chi(T) - \chi(S \cap T) \quad \text{(under hypotheses)}$$
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Challenge Find a general definition of 'size', including these and other examples.

One answer The magnitude of an enriched category.

1. The cardinality of a colimit

The problem

Some familiar formulas for cardinalities of finite sets:

- Inclusion-exclusion formula:

$$|S \cup T| = |S| + |T| - |S \cap T|$$

- Orbits of a group acting freely:

$$|S/G| = |S| \,/\, |G| \,.$$

Problem Given a finite category $\mathbf{A}$, are there 'weights' $\big(w(a)\big)_{a \in \mathbf{A}}$ such that

$$|\operatorname{colim} X| = \sum_{a \in \mathbf{A}} w(a)\, |X(a)|$$

for any functor $X \colon \mathbf{A} \to \mathbf{FinSet}$?

Obviously not for an *arbitrary* X, but maybe under restrictions on X...

A solution

Given a finite category $\mathbf{A}$, write $Z_{\mathbf{A}}$ for the $\mathrm{ob}\,\mathbf{A} \times \mathrm{ob}\,\mathbf{A}$ matrix with entries

$$Z_{\mathbf{A}}(a, b) = |\mathbf{A}(a, b)|\,.$$

Definition Let Z be a matrix. A weighting on Z is a column vector $\mathbf{w}$ such that $Z\mathbf{w} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

E.g. A weighting on $Z_{\mathbf{A}}$ is a family $\big(w(a)\big)_{a \in \mathbf{A}}$ in $\mathbb{Q}$ such that

$$\sum_b |\mathbf{A}(a, b)|\, w(b) = 1$$

for all $a \in \mathbf{A}$.

Theorem Let $\mathbf{A}$ be a finite category and $\mathbf{w}$ a weighting on $Z_{\mathbf{A}}$. Then

$$|\mathrm{colim}\, X| = \sum_{a \in \mathbf{A}} w(a)\, |X(a)|$$

for any functor $X \colon \mathbf{A} \to \mathbf{FinSet}$ that is a coproduct of representables.

Examples

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Examples

- $\mathbf{A}$ discrete: unique weighting is $w(a) \equiv 1$, and Theorem gives $|\coprod_a X(a)| = \sum_a |X(a)|$.
- $\mathbf{A} = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$: unique weighting is $\begin{pmatrix} -1 & 1 \\ & \\ 1 & \end{pmatrix}$, and Theorem gives the inclusion-exclusion formula.
- $\mathbf{A} = G$ (one-object category): unique weighting is $1/\mathrm{order}(G)$, and Theorem gives cardinality formula for free group action.

Remarks The theory connects to Möbius–Rota inversion for posets.

Ponto and Shulman have a more sophisticated version of the theorem.

What if . . . ?

Theorem Let $\mathbf{A}$ be a finite category and $\mathbf{w}$ a weighting on $Z_{\mathbf{A}}$. Then

$$|\operatorname{colim} X| = \sum_{a \in \mathbf{A}} w(a)\,|X(a)|$$

for any functor $X \colon \mathbf{A} \to \mathbf{FinSet}$ that is a coproduct of representables.

Question What if we put the constant functor $X = \Delta 1$ into the formula?

Usually $\Delta 1$ is *not* a coproduct of representables, and equality fails.

But the right-hand side still calculates *something*. It's a number associated with the category $\mathbf{A}$:

$$\sum_{a \in \mathbf{A}} w(a).$$

E.g. If $\mathbf{A}$ is discrete then $w(a) \equiv 1$, so $\sum w(a)$ is the number of objects.

What does $\sum w(a)$ mean in general?

2. *The magnitude of a category*

The magnitude of a matrix

Definition Let Z be a matrix. Suppose both Z and Z^{T} admit a weighting. The magnitude of Z is the total weight

$$|Z| = \sum_i w_i,$$

where $\mathbf{w} = (w_i)$ is any weighting on Z.

(Easy lemma: this is independent of the weighting chosen.)

Important special case If Z is invertible then it has a unique weighting, and

$$|Z| = \sum_{i,j} \left(Z^{-1}\right)_{ij}.$$

The magnitude of a category

Let $\mathbf{A}$ be a finite category. The magnitude (or Euler characteristic) of $\mathbf{A}$ is

$$|\mathbf{A}| = |Z_{\mathbf{A}}| \in \mathbb{Q}.$$

It is defined as long as $Z_{\mathbf{A}}$ and $Z_{\mathbf{A}}^{\mathsf{T}}$ both admit weightings over $\mathbb{Q}$.

Examples

- If $\mathbf{A}$ is discrete then $|\mathbf{A}| = \text{cardinality}(\text{ob}\,\mathbf{A})$.
- If $\mathbf{A}$ is a monoid M (as one-object category) then $|A| = 1/\text{order}(M)$.
- If $\mathbf{A}$ is a groupoid then

  $$|\mathbf{A}| = \sum_a 1/\text{order}(\text{Aut}(a)),$$

  where the sum is over representatives of iso classes: the groupoid cardinality. ('E.g.' $|\text{finite sets \& bijections}| = e = 2.718\ldots$.)
- If $\mathbf{A} = (\bullet \rightrightarrows \bullet)$ then

  $$Z_{\mathbf{A}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad Z_{\mathbf{A}}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

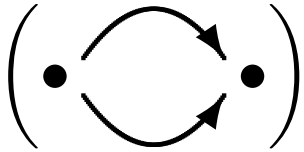
  and $|\mathbf{A}| = 1 + (-2) + 0 + 1 = 0$.

Relation to topological Euler characteristic

Recall Every small category $\mathbf{A}$ has a classifying space $B\mathbf{A} \in \mathbf{Top}$.

Theorem Let $\mathbf{A}$ be a category whose nerve has only finitely many nondegenerate simplices. Then

$$\chi(B\mathbf{A}) = |\mathbf{A}|\,.$$

E.g. If $\mathbf{A} =$  then $B\mathbf{A} = S^1$ and $\chi(S^1) = 0 = |\mathbf{A}|$.

Other theorems connect magnitude of categories to Euler characteristic of manifolds — and more generally, orbifolds (whose Euler characteristics are usually $\notin \mathbb{Z}$).

Theorems on magnitude of categories

- If $\mathbf{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{B}$ and each has well-defined magnitude then $|\mathbf{A}| = |\mathbf{B}|$.
- Corollary: if $\mathbf{A}$ has an initial or terminal object then $|\mathbf{A}| = 1$.
- $|\prod_i \mathbf{A}_i| = \prod_i |\mathbf{A}_i|$ and $|\coprod_i \mathbf{A}_i| = \sum_i |\mathbf{A}_i|$ (plus similar, more sophisticated, results).

3. The magnitude of an enriched category

The idea

To define the magnitude of a finite category $\mathbf{A}$, we used the matrix $Z_{\mathbf{A}}$ with entries

$$Z_{\mathbf{A}}(a, b) = |\mathbf{A}(a, b)| \,.$$

The right-hand side is the cardinality of a finite set.

So:

starting from the notion of the size of an *object of* **FinSet**,

we obtained a notion of the size of a *category enriched in* **FinSet**.

Idea: Do the same with an arbitrary monoidal category in place of **FinSet**.

The definition

Let $\mathscr{V}$ be a monoidal category and k a (semi)ring.

Let

$$|\cdot| : \frac{\operatorname{ob}\mathscr{V}}{\cong} \to k$$

be a monoid homomorphism (so $|x \otimes y| = |x|\,|y|$ and $|I| = 1$).

Given a $\mathscr{V}$-category $\mathbf{A}$ with finitely many objects, write $Z_{\mathbf{A}}$ for the $\operatorname{ob}\mathbf{A} \times \operatorname{ob}\mathbf{A}$ matrix with entries

$$Z_{\mathbf{A}}(a, b) = |\mathbf{A}(a, b)|\,.$$

The magnitude of $\mathbf{A}$ is $|\mathbf{A}| = |Z_{\mathbf{A}}| \in k$ (if defined).

E.g. Take $\mathscr{V} = \mathbf{FinSet}$, $k = \mathbb{Q}$, and $|\cdot| = \mathrm{card}$: then we recover the definition of the magnitude of a finite category.

The magnitude of a linear category

Let F be a field and $\mathscr{V} = \mathbf{FDVect}_F$. For $X \in \mathscr{V}$, put $|X| = \dim X \in \mathbb{Q}$.

Get notion of the magnitude $|\mathbf{A}| \in \mathbb{Q}$ of a finite linear category $\mathbf{A}$.

Example Let E be an associative algebra over F.

An important linear category associated with E is

$$\mathbf{IP}(E) = (\text{indecomposable projective } E\text{-modules}) \subseteq_{\text{full}} E\text{-}\mathbf{Mod}.$$

Theorem (with Chuang and King) Under finiteness hypotheses,

$$|\mathbf{IP}(E)| = \sum_{n=0}^{\infty} (-1)^n \dim \operatorname{Ext}_E^n(S, S),$$

where S is the direct sum of the simple E-modules.

(The matrix $Z_{\mathbf{IP}(E)}$ is known as the 'Cartan matrix' of E.
The sum $\sum (-1)^n \cdots$ is known as the 'Euler form' of E at (S, S).)

The magnitude of a metric space

Let $\mathscr{V} = ([0, \infty], \geq, +, 0)$, so that metric spaces are $\mathscr{V}$-categories.

Define $|\cdot| : [0, \infty] \to \mathbb{R}$ by $|x| = e^{-x}$.

(Why? So that $|x + y| = |x|\,|y|$ and $|0| = 1$.)

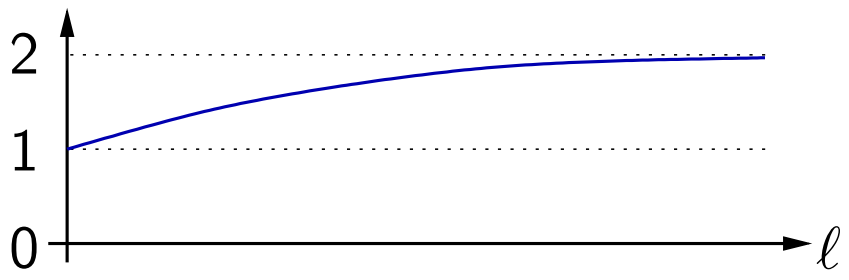
Get notion of the magnitude $|A| \in \mathbb{R}$ of a finite metric space A.

Explicitly: to compute the magnitude of a metric space $A = \{a_1, \ldots, a_n\}$:

- write down the $n \times n$ matrix with (i, j)-entry $e^{-d(a_i, a_j)}$
- invert it
- add up all n^2 entries.

The magnitude of a finite metric space: first examples

- $|\emptyset| = 0$.
- $|\bullet| = 1$.
- $|\bullet \overset{\leftarrow \ \ell \ \rightarrow}{} \bullet| =$ sum of entries of $\begin{pmatrix} e^{-0} & e^{-\ell} \\ e^{-\ell} & e^{-0} \end{pmatrix}^{-1} = \dfrac{2}{1 + e^{-\ell}}$

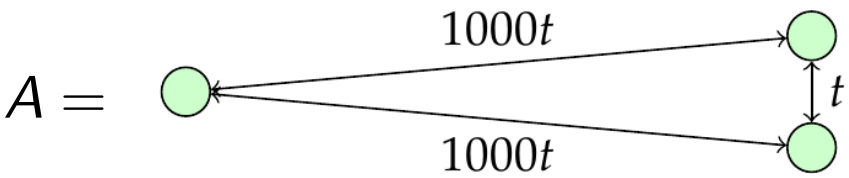


- If $d(a, b) = \infty$ for all $a \neq b$ then $|A| =$ cardinality(A).

Slogan: Magnitude is the 'effective number of points'

Example: a 3-point space (Simon Willerton)

Take the 3-point space



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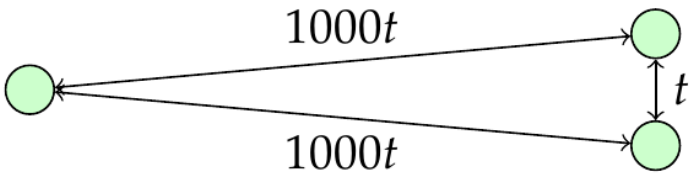
$A =$ 

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- When t is moderate, A looks like a 2-point space.

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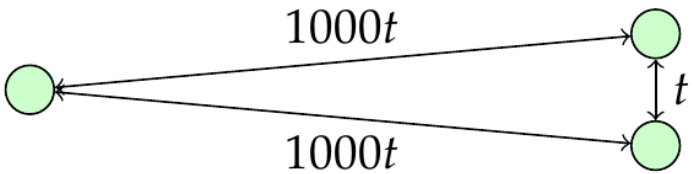
$A =$ 

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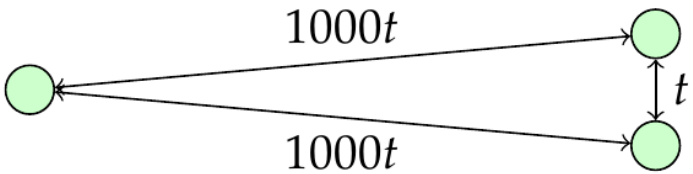
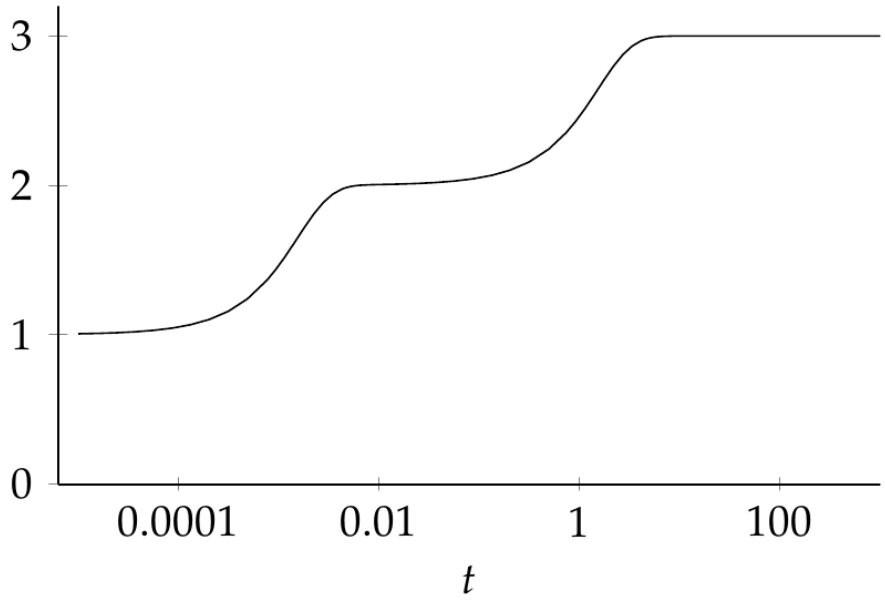
$A =$ 

- When t is small, A looks like a 1-point space.
- When t is moderate, A looks like a 2-point space.
- When t is large, A looks like a 3-point space.

Indeed, the magnitude of A as a function of t is:



Magnitude functions

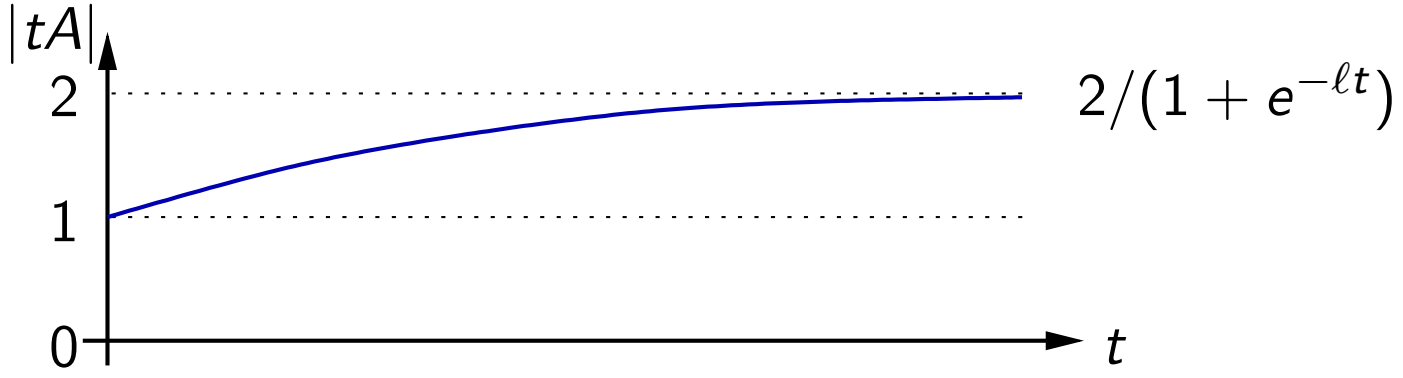
Magnitude assigns to each metric space not just a *number*, but a *function*.

For $t > 0$, write tA for A scaled up by a factor of t.

The magnitude function of a metric space A is the partial function

$$\begin{array}{ccc} (0, \infty) & \to & \mathbb{R} \\ t & \mapsto & |tA| \,. \end{array}$$

E.g.: the magnitude function of $A = (\bullet \overset{\leftarrow \ \ell \ \rightarrow}{} \bullet)$ is



A magnitude function has only finitely many singularities (none if $A \subseteq \mathbb{R}^n$).

It is increasing for $t \gg 0$, and $\lim_{t \to \infty} |tA| = \text{cardinality}(A)$.

The magnitude of a compact metric space

In principle, magnitude is only defined for enriched categories *with finitely many objects* — here, *finite* metric spaces.

Can the definition be extended to, say, compact metric spaces?

Theorem (Mark Meckes)
All sensible ways of extending the definition of magnitude from finite metric spaces to compact 'positive definite' spaces are equivalent.

Proof Uses functional analysis.

Definition of 'positive definite' omitted here, but includes all subspaces of $\mathbb{R}^n$ with Euclidean or ℓ^1 (taxicab) metric, and many other common spaces.

The magnitude of a compact positive definite space A is

$$|A| = \sup\{|B| \ : \ \text{finite } B \subseteq A\}.$$

Magnitude of a compact space: examples

E.g. Line segment: $|t[0,\ell]| = 1 + \frac{1}{2}\ell \cdot t$.

Sample theorem Let $A \subseteq \mathbb{R}^2$ be a convex body with the ℓ^1 (taxicab) metric. Then

$$|tA| = \chi(A) + \tfrac{1}{4}\text{perimeter}(A) \cdot t + \tfrac{1}{4}\text{area}(A) \cdot t^2.$$

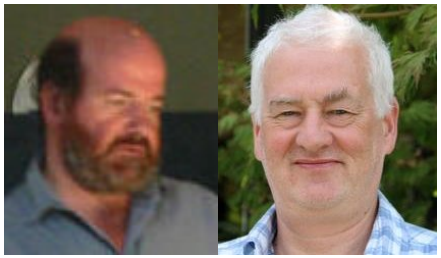
There's a similar theorem in higher dimensions.

Magnitude encodes geometric information

Let A be a compact subset of $\mathbb{R}^n$, with Euclidean metric.

Theorem (Meckes) From the magnitude function of A, you can recover the Minkowski dimension of A.

Proof Uses a deep theorem from potential analysis, plus the notion of maximum diversity.



Theorem (Barceló and Carbery) From the magnitude function of A, you can recover the volume of A.

Proof Uses PDEs and Fourier analysis.

Theorem (Gimperlein and Goffeng) From the magnitude function of A, you can recover the surface area of A.

(Needs n odd and some regularity hypotheses.)

Proof Uses heat trace asymptotics (techniques related to the heat equation proof of the Atiyah–Singer index theorem).

Inclusion-exclusion for magnitude

Theorem (Gimperlein and Goffeng) Let $A, B \subseteq \mathbb{R}^n$, subject to technical hypotheses. Then

$$|t(A \cup B)| + |t(A \cap B)| - |tA| - |tB| \to 0$$

as $t \to \infty$.

Magnitude of metric spaces doesn't *literally* obey inclusion-exclusion, as that would make it trivial.

But it *asymptotically* does.

Digression: (bio)diversity

Digression: (bio)diversity

Conceptual question Given an ecological community, consisting of individuals grouped into species, how can we reasonably quantify its 'diversity'?

Simplest answer Count the number n of species present.

(Mathematically: cardinality of a finite set.)

Better answer Use the relative abundance distribution $\mathbf{p} = (p_1, \ldots, p_n)$ of species.

For any choice of parameter $q \in \mathbb{R}^+$, can quantify diversity as

$$D_q(\mathbf{p}) = \left(\sum_i p_i^q \right)^{1/(1-q)}.$$

(E.g. if $\mathbf{p} = (1/n, \ldots, 1/n)$ then $D_q(\mathbf{p}) = n$.)

(Mathematically: $\sim$entropy of a probability distribution on a finite set.)

Digression: (bio)diversity

Even better answer Also use the matrix Z of similarities between species.

For any choice of parameter $q \in \mathbb{R}^+$, can quantify diversity as

$$D_q^Z(\mathbf{p}) = \left(\sum_i p_i (Z\mathbf{p})_i^{q-1} \right)^{1/(1-q)}.$$

The formula is not important here. But...

Discovery (with Christina Cobbold) Most of the biodiversity measures most commonly used in ecology are special cases of D_q^Z.

(Mathematically: $\sim$entropy of a probability distribution on a finite metric space.)

Digression: (bio)diversity

The maximization problem

Fix a list of species, with known similarity matrix Z.

What is the maximum diversity that can be achieved by varying the species abundances? I.e., what is $\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$?

In principle, the answer depends on the parameter q.

Theorem (with Mark Meckes) The answer is independent of q.

So, $\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$ is a canonical number associated with the matrix Z — the maximum diversity $D_{\max}(Z)$ of Z.

Fact $D_{\max}(Z)$ is the magnitude of some submatrix of Z.

Conclusion: Magnitude is closely related to maximum diversity.

End of digression

. . . back to magnitude of $\mathscr{V}$-categories

The magnitude of a graph

Any graph A can be viewed as a metric space:

- points are vertices
- distances are shortest path-lengths (which are integers!).

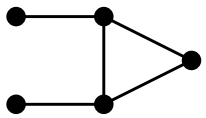
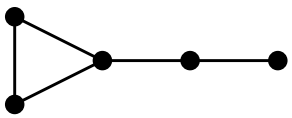
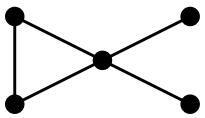
The magnitude of the graph A is the magnitude of this metric space.

Fact The magnitude function $t \mapsto |tA|$ is a *rational function* over $\mathbb{Z}$ of the formal variable $x = e^{-t}$.

It can also be expanded as a *power series* in x over $\mathbb{Z}$.

The magnitude of a graph: examples and theorems

Examples

$\Big|$  $\Big| \;=\; \Big|$  $\Big| = \Big|$  $\Big| = \dfrac{5+5x-4x^2}{(1+x)(1+2x)}$

$$= \; 5 - 10x + 16x^2 - 28x^3 + \cdots$$

Sample theorems:

- $|A \otimes B| = |A| \cdot |B|$, where $\otimes$ is a certain graph product
- $|A \cup B| = |A| + |B| - |A \cap B|$, under quite strict hypotheses
- Graph magnitude has other invariance properties shared with the Tutte polynomial.

Magnitude of other enriched categories

Magnitude of n-categories

- Start with the notion of the size (cardinality) of a finite set.
- Taking $\mathscr{V} = \mathbf{FinSet}$, automatically get notion of the size (magnitude) of a finite 1-category.
- Taking $\mathscr{V} = \mathbf{FinCat}$, automatically get notion of the size (magnitude) of a finite 2-category.
- ...
- Automatically get notion of the size (magnitude) of a finite n-category ($n < \infty$).

Almost nothing is known about this!

And what is the magnitude of an ∞-category?

Also What about other bases $\mathscr{V}$ of enrichment?

4. Where's the category theory?

Overview


categories
magnitude
graphs
posets
metric spaces
groupoids
diversity
n-cats
?
linear cats

Overview


magnitude
homology


categories
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5. Magnitude homology: a sketch

Two perspectives on Euler characteristic

So far: Euler characteristic has been treated as an analogue of cardinality.

Alternatively: Given any homology theory H_* of any kind of object A, can define

$$\chi(A) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(A).$$

Note:

- $\chi(A)$ is a *number*
- $H_*(A)$ is an *algebraic structure*, and functorial in A.

In this sense, homology is a categorification of Euler characteristic.

The homology of an ordinary category

Let $\mathbf{A}$ be a small category.

Its nerve $N\mathbf{A}$ is a simplicial set.

Form the associated chain complex $C_*(\mathbf{A})$ in the usual way.

The homology $H_*(\mathbf{A})$ of $\mathbf{A}$ is the homology of $C_*(\mathbf{A})$.

Theorem $H_*(\mathbf{A}) = H_*(B\mathbf{A})$.

Hence

$$\sum_{n=0}^{\infty}(-1)^n \operatorname{rank} H_n(\mathbf{A}) = \sum_{n=0}^{\infty}(-1)^n \operatorname{rank} H_n(B\mathbf{A}) = \chi(B\mathbf{A}) = |\mathbf{A}|.$$

Goal For a $\mathscr{V}$-category $\mathbf{A}$, define a 'homology' $H_*(\mathbf{A})$ in such a way that

$$\sum_{n=0}^{\infty}(-1)^n \operatorname{rank} H_n(\mathbf{A}) = |\mathbf{A}|\,.$$

It can be done!

The magnitude homology of a graph

Richard Hepworth and Simon Willerton defined the magnitude homology of a graph A.

(Definition omitted here.)

Features:

- It's a *graded* homology theory, i.e. each $H_n(A)$ is a *graded* abelian group.
- Hence $\chi(A) = \sum(-1)^n \operatorname{rank} H_n(A)$ is a *sequence* of integers.
- Viewing this sequence as a power series over $\mathbb{Z}$, it is exactly the magnitude of A.
  So: magnitude homology categorifies magnitude.
- The formulas for $|A \otimes B|$ and $|A \cup B|$ can be categorified to give Künneth and Mayer–Vietoris theorems.
- Magnitude homology can distinguish between graphs that mere magnitude cannot.

The magnitude homology of an enriched category

Let $\mathscr{V}$ be a monoidal category.

Mike Shulman gave a general definition of the magnitude homology $H_*(\mathbf{A})$ of a $\mathscr{V}$-category $\mathbf{A}$.

(Definition omitted here.)

Features:

- It generalizes both homology of ordinary categories and magnitude homology of graphs.
- The Euler characteristic of the magnitude homology $H_*(\mathbf{A})$ is the magnitude $|\mathbf{A}|$ (in a suitably formal sense).
  So: magnitude homology categorifies magnitude.
- The general definition is a kind of Hochschild homology.
- There's an accompanying cohomology theory.

The magnitude homology of a metric space

In particular, the general definition gives a homology theory of metric spaces.

It's a genuinely *metric* homology theory — not just topological.

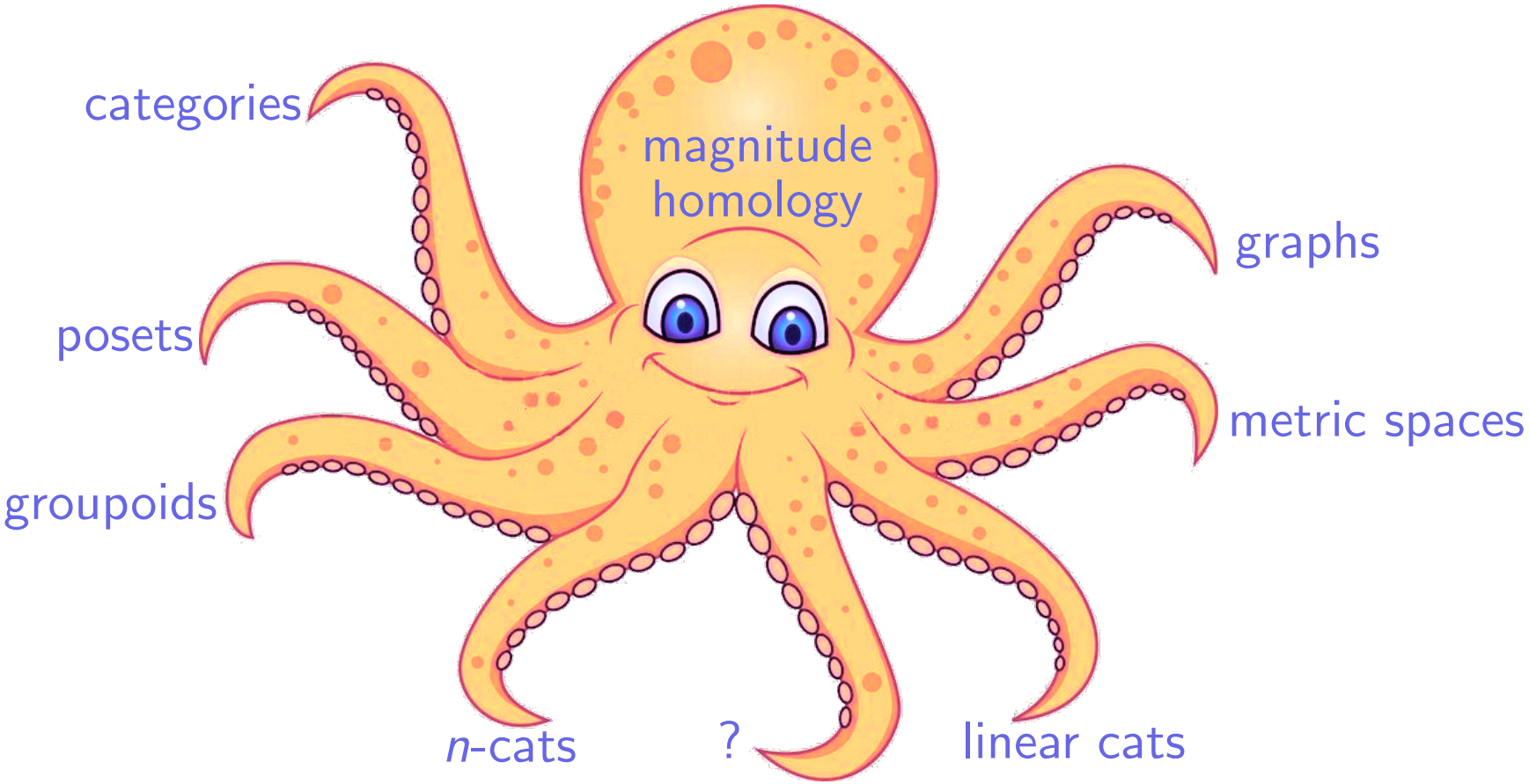
Sample theorem For compact $A \subseteq \mathbb{R}^n$,

$$H_1(A) = 0 \iff A \text{ is convex.}$$

Very recent result of Nina Otter (arXiv paper last Wednesday):

magnitude homology is related to (but different from!) persistent homology.

Summary


categories
magnitude
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n-cats
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linear cats


categories
magnitude
graphs
posets
metric spaces
groupoids
diversity
n-cats
?
linear cats

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