Magnitude

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The idea
For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies
  \[ |S \cup T| = |S| + |T| - |S \cap T| \]
  \[ |S \times T| = |S| \times |T|. \]

- Subsets of \( \mathbb{R}^n \) have volume. It satisfies
  \[ \text{vol}(S \cup T) = \text{vol}(S) + \text{vol}(T) - \text{vol}(S \cap T) \]
  \[ \text{vol}(S \times T) = \text{vol}(S) \times \text{vol}(T). \]

- Topological spaces have Euler characteristic. It satisfies
  \[ \chi(S \cup T) = \chi(S) + \chi(T) - \chi(S \cap T) \quad \text{(under hypotheses)} \]
  \[ \chi(S \times T) = \chi(S) \times \chi(T). \]

Stephen Schanuel:
Euler characteristic is the topological analogue of cardinality.
The idea

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Challenge Find a general definition of ‘size’, including these and other examples.

One answer The magnitude of an enriched category.
1. The cardinality of a colimit
The problem

Some familiar formulas for cardinalities of finite sets:

- Inclusion-exclusion formula:
  \[ |S \cup T| = |S| + |T| - |S \cap T| \]

- Orbits of a group acting freely:
  \[ |S/G| = |S| / |G| \]

Problem Given a finite category \( A \), are there ‘weights’ \( (w(a))_{a \in A} \) such that

\[ |\text{colim} \ X| = \sum_{a \in A} w(a) |X(a)| \]

for any functor \( X : A \to \text{FinSet} \)?

Obviously not for an \textit{arbitrary} \( X \), but maybe under restrictions on \( X \ldots \)
A solution

Given a finite category $\mathbf{A}$, write $Z_\mathbf{A}$ for the $\text{ob} \mathbf{A} \times \text{ob} \mathbf{A}$ matrix with entries

$$Z_\mathbf{A}(a, b) = |A(a, b)|.$$

Definition Let $Z$ be a matrix. A weighting on $Z$ is a column vector $w$ such that $Zw = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

E.g. A weighting on $Z_\mathbf{A}$ is a family $(w(a))_{a \in \mathbf{A}}$ in $\mathbb{Q}$ such that

$$\sum_{b} |A(a, b)| w(b) = 1$$

for all $a \in \mathbf{A}$.

Theorem Let $\mathbf{A}$ be a finite category and $w$ a weighting on $Z_\mathbf{A}$. Then

$$|\text{colim} X| = \sum_{a \in \mathbf{A}} w(a) |X(a)|$$

for any functor $X : \mathbf{A} \to \text{FinSet}$ that is a coproduct of representables.
Theorem Let $A$ be a finite category and $w$ a weighting on $Z_A$. Then

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Examples

- $A$ discrete: unique weighting is $w(a) \equiv 1$, and Theorem gives $|\coprod_a X(a)| = \sum_a |X(a)|$.
- $A = \bullet \rightarrow \bullet \downarrow \bullet$: unique weighting is $\begin{pmatrix} -1 & 1 \\ 1 & \end{pmatrix}$, and Theorem gives the inclusion-exclusion formula.
- $A = G$ (one-object category): unique weighting is $1 / \text{order}(G)$, and Theorem gives cardinality formula for free group action.

Remarks  The theory connects to Möbius–Rota inversion for posets.

Ponto and Shulman have a more sophisticated version of the theorem.
What if . . .?

Theorem Let $A$ be a finite category and $w$ a weighting on $Z_A$. Then

$$|\text{colim } X| = \sum_{a \in A} w(a) |X(a)|$$

for any functor $X : A \to \text{FinSet}$ that is a coproduct of representables.

Question What if we put the constant functor $X = \Delta 1$ into the formula?

Usually $\Delta 1$ is not a coproduct of representables, and equality fails. But the right-hand side still calculates something. It’s a number associated with the category $A$:

$$\sum_{a \in A} w(a).$$

E.g. If $A$ is discrete then $w(a) \equiv 1$, so $\sum w(a)$ is the number of objects.

What does $\sum w(a)$ mean in general?
2. The magnitude of a category
The magnitude of a matrix

Definition  Let $Z$ be a matrix. Suppose both $Z$ and $Z^T$ admit a weighting. The magnitude of $Z$ is the total weight

$$|Z| = \sum_{i} w_i,$$

where $\mathbf{w} = (w_i)$ is any weighting on $Z$.

(Easy lemma: this is independent of the weighting chosen.)

Important special case  If $Z$ is invertible then it has a unique weighting, and

$$|Z| = \sum_{i,j} (Z^{-1})_{ij}.$$
The magnitude of a category

Let \( \mathbf{A} \) be a finite category. The magnitude (or Euler characteristic) of \( \mathbf{A} \) is

\[
|\mathbf{A}| = |\mathcal{Z}_\mathbf{A}| \in \mathbb{Q}.
\]

It is defined as long as \( \mathcal{Z}_\mathbf{A} \) and \( \mathcal{Z}_\mathbf{A}^* \) both admit weightings over \( \mathbb{Q} \).

Examples

- If \( \mathbf{A} \) is discrete then \( |\mathbf{A}| = \text{cardinality}(\text{ob} \mathbf{A}) \).
- If \( \mathbf{A} \) is a monoid \( M \) (as one-object category) then \( |\mathbf{A}| = 1/\text{order}(M) \).
- If \( \mathbf{A} \) is a groupoid then

\[
|\mathbf{A}| = \sum_a 1/\text{order}(\text{Aut}(a)),
\]

where the sum is over representatives of iso classes: the groupoid cardinality. (‘E.g.’ \( |\text{finite sets & bijections}| = e = 2.718 \ldots \))
- If \( \mathbf{A} = (\bullet \Rightarrow \bullet) \) then

\[
\mathcal{Z}_\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{Z}_\mathbf{A}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},
\]

and \( |\mathbf{A}| = 1 + (-2) + 0 + 1 = 0. \)
Relation to topological Euler characteristic

Recall Every small category $A$ has a classifying space $BA \in \text{Top}$.

Theorem Let $A$ be a category whose nerve has only finitely many nondegenerate simplices. Then

$$\chi(BA) = |A|.$$  

E.g. If $A = \begin{array}{c} \bullet \\ \circlearrowright \\ \bullet \end{array}$ then $BA = S^1$ and $\chi(S^1) = 0 = |A|$.  

Other theorems connect magnitude of categories to Euler characteristic of manifolds — and more generally, orbifolds (whose Euler characteristics are usually $\not\in \mathbb{Z}$).
Theorems on magnitude of categories

- If $A \leftrightarrow B$ and each has well-defined magnitude then $|A| = |B|$.

- Corollary: if $A$ has an initial or terminal object then $|A| = 1$.

- $\prod_i A_i = \prod_i |A_i|$ and $\bigsqcup_i A_i = \sum_i |A_i|$ (plus similar, more sophisticated, results).
3. The magnitude of an enriched category
The idea

To define the magnitude of a finite category $\mathbf{A}$, we used the matrix $Z_\mathbf{A}$ with entries

$$Z_\mathbf{A}(a, b) = |\mathbf{A}(a, b)|.$$ 

The right-hand side is the cardinality of a finite set.

So:

starting from the notion of the size of an object of $\text{FinSet}$, we obtained a notion of the size of a category enriched in $\text{FinSet}$.

Idea: Do the same with an arbitrary monoidal category in place of $\text{FinSet}$. 
The definition

Let \( \mathcal{V} \) be a monoidal category and \( k \) a (semi)ring. Let

\[ |\cdot| : \text{ob} \mathcal{V} \rightarrow k \]

be a monoid homomorphism (so \(|x \otimes y| = |x| |y|\) and \(|I| = 1\)). Given a \( \mathcal{V} \)-category \( A \) with finitely many objects, write \( Z_A \) for the \( \text{ob} A \times \text{ob} A \) matrix with entries

\[ Z_A(a, b) = |A(a, b)|. \]

The magnitude of \( A \) is \(|A| = |Z_A| \in k\) (if defined).

E.g. Take \( \mathcal{V} = \text{FinSet} \), \( k = \mathbb{Q} \), and \(|\cdot| = \text{card}\): then we recover the definition of the magnitude of a finite category.
The magnitude of a linear category

Let $F$ be a field and $\mathcal{V} = \text{FDVect}_F$. For $X \in \mathcal{V}$, put $|X| = \dim X \in \mathbb{Q}$.

Get notion of the magnitude $|A| \in \mathbb{Q}$ of a finite linear category $A$.

**Example** Let $E$ be an associative algebra over $F$.

An important linear category associated with $E$ is

$$\text{IP}(E) = (\text{indecomposable projective } E\text{-modules}) \subset \text{full } E\text{-Mod}.$$

**Theorem (with Chuang and King)** Under finiteness hypotheses,

$$|\text{IP}(E)| = \sum_{n=0}^{\infty} (-1)^n \dim \text{Ext}_E^n(S, S),$$

where $S$ is the direct sum of the simple $E$-modules.

(The matrix $Z_{\text{IP}(E)}$ is known as the ‘Cartan matrix’ of $E$.
The sum $\sum (-1)^n \cdots$ is known as the ‘Euler form’ of $E$ at $(S, S)$.)
The magnitude of a metric space

Let $\mathcal{V} = ([0, \infty], \geq, +, 0)$, so that metric spaces are $\mathcal{V}$-categories.

Define $|\cdot| : [0, \infty] \to \mathbb{R}$ by $|x| = e^{-x}$.

(Why? So that $|x + y| = |x||y|$ and $|0| = 1$.)

Get notion of the magnitude $|A| \in \mathbb{R}$ of a finite metric space $A$.

Explicitly: to compute the magnitude of a metric space $A = \{a_1, \ldots, a_n\}$:

- write down the $n \times n$ matrix with $(i, j)$-entry $e^{-d(a_i, a_j)}$
- invert it
- add up all $n^2$ entries.
The magnitude of a finite metric space: first examples

- $|\emptyset| = 0$.
- $|\bullet| = 1$.
- $|\overset{\leftarrow}{\bullet} \rightarrow \bullet| = \text{sum of entries of } \begin{pmatrix} e^{-0} & e^{-\ell} \\ e^{-\ell} & e^{-0} \end{pmatrix}^{-1} = \frac{2}{1 + e^{-\ell}}$

If $d(a, b) = \infty$ for all $a \neq b$ then $|A| = \text{cardinality}(A)$.

**Slogan:** Magnitude is the ‘effective number of points’
Example: a 3-point space (Simon Willerton)

Take the 3-point space

\[ A = \]

- When \( t \) is small, \( A \) looks like a 1-point space.

- When \( t \) is moderate, \( A \) looks like a 2-point space.

- When \( t \) is large, \( A \) looks like a 3-point space.
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Indeed, the magnitude of \( A \) as a function of \( t \) is:
Magnitude functions

Magnitude assigns to each metric space not just a \textit{number}, but a \textit{function}. For \( t > 0 \), write \( tA \) for \( A \) scaled up by a factor of \( t \).

The magnitude function of a metric space \( A \) is the partial function

\[
(0, \infty) \to \mathbb{R} \\
\quad t \mapsto |tA|.
\]

E.g.: the magnitude function of \( A = (\bullet \leftarrow \ell \rightarrow \bullet) \) is \( 2/(1 + e^{-\ell t}) \).

A magnitude function has only finitely many singularities (none if \( A \subseteq \mathbb{R}^n \)). It is increasing for \( t \gg 0 \), and \( \lim_{t \to \infty} |tA| = \text{cardinality}(A) \).
The magnitude of a compact metric space

In principle, magnitude is only defined for enriched categories *with finitely many objects* — here, *finite* metric spaces.

Can the definition be extended to, say, compact metric spaces?

**Theorem (Mark Meckes)**

All sensible ways of extending the definition of magnitude from finite metric spaces to compact ‘positive definite’ spaces are equivalent.

**Proof** Uses functional analysis.

Definition of ‘positive definite’ omitted here, but includes all subspaces of $\mathbb{R}^n$ with Euclidean or $\ell^1$ (taxicab) metric, and many other common spaces.

The **magnitude** of a compact positive definite space $A$ is

$$|A| = \sup\{|B| : \text{finite } B \subseteq A\}.$$
Magnitude of a compact space: examples

E.g. Line segment: \(|t[0, \ell]| = 1 + \frac{1}{2}\ell \cdot t.\)

Sample theorem Let \(A \subseteq \mathbb{R}^2\) be a convex body with the \(\ell^1\) (taxicab) metric. Then
\[
|tA| = \chi(A) + \frac{1}{4}\text{perimeter}(A) \cdot t + \frac{1}{4}\text{area}(A) \cdot t^2.
\]

There’s a similar theorem in higher dimensions.
Magnitude encodes geometric information

Let $A$ be a compact subset of $\mathbb{R}^n$, with Euclidean metric.

**Theorem (Meckes)** From the magnitude function of $A$, you can recover the Minkowski dimension of $A$.

*Proof* Uses a deep theorem from potential analysis, plus the notion of maximum diversity.

**Theorem (Barceló and Carbery)** From the magnitude function of $A$, you can recover the volume of $A$.

*Proof* Uses PDEs and Fourier analysis.

**Theorem (Gimperlein and Goffeng)** From the magnitude function of $A$, you can recover the surface area of $A$.

(Needs $n$ odd and some regularity hypotheses.)

*Proof* Uses heat trace asymptotics (techniques related to the heat equation proof of the Atiyah–Singer index theorem).
Inclusion-exclusion for magnitude

Theorem (Gimperlein and Goffeng) Let $A, B \subseteq \mathbb{R}^n$, subject to technical hypotheses. Then

$$|t(A \cup B)| + |t(A \cap B)| - |tA| - |tB| \to 0$$

as $t \to \infty$.

Magnitude of metric spaces doesn’t literally obey inclusion-exclusion, as that would make it trivial.

But it asymptotically does.
Digression: (bio)diversity
Digression: (bio)diversity

Conceptual question Given an ecological community, consisting of individuals grouped into species, how can we reasonably quantify its ‘diversity’?

Simplest answer Count the number $n$ of species present.
(Mathematically: cardinality of a finite set.)

Better answer Use the relative abundance distribution $\mathbf{p} = (p_1, \ldots, p_n)$ of species.

For any choice of parameter $q \in \mathbb{R}^+$, can quantify diversity as

$$D_q(\mathbf{p}) = 
\left( \sum_{i} p_i^q \right)^{1/(1-q)}.
$$

(E.g. if $\mathbf{p} = (1/n, \ldots, 1/n)$ then $D_q(\mathbf{p}) = n$.)
(Mathematically: $\sim$entropy of a probability distribution on a finite set.)
Digression: (bio)diversity

Even better answer Also use the matrix $Z$ of similarities between species. For any choice of parameter $q \in \mathbb{R}^+$, can quantify diversity as

$$D^Z_q(p) = \left( \sum_i p_i (Zp)_i^{q-1} \right)^{1/(1-q)}.$$  

The formula is not important here. But…

Discovery (with Christina Cobbold) Most of the biodiversity measures most commonly used in ecology are special cases of $D^Z_q$.

(Mathematically: $\sim$entropy of a probability distribution on a finite metric space.)
Digression: (bio)diversity

The maximization problem
Fix a list of species, with known similarity matrix $Z$.

What is the maximum diversity that can be achieved by varying the species abundances? I.e., what is $\sup_{\mathbf{p}} D^Z_{q}(\mathbf{p})$?

In principle, the answer depends on the parameter $q$.

Theorem (with Mark Meckes) The answer is independent of $q$.

So, $\sup_{\mathbf{p}} D^Z_{q}(\mathbf{p})$ is a canonical number associated with the matrix $Z$ — the maximum diversity $D_{\text{max}}(Z)$ of $Z$.

Fact $D_{\text{max}}(Z)$ is the magnitude of some submatrix of $Z$.

Conclusion: Magnitude is closely related to maximum diversity.
End of digression

...back to magnitude of \( \mathcal{V} \)-categories
The magnitude of a graph

Any graph $A$ can be viewed as a metric space:

- points are vertices
- distances are shortest path-lengths (which are integers!).

The magnitude of the graph $A$ is the magnitude of this metric space.

**Fact** The magnitude function $t \mapsto |tA|$ is a *rational function* over $\mathbb{Z}$ of the formal variable $x = e^{-t}$.

It can also be expanded as a *power series* in $x$ over $\mathbb{Z}$.
The magnitude of a graph: examples and theorems

Examples

\[ \left| \begin{array}{c}
\text{\includegraphics{example1}}
\end{array} \right| = \left| \begin{array}{c}
\text{\includegraphics{example2}}
\end{array} \right| = \left| \begin{array}{c}
\text{\includegraphics{example3}}
\end{array} \right| = \frac{5 + 5x - 4x^2}{(1 + x)(1 + 2x)} \]

\[ = 5 - 10x + 16x^2 - 28x^3 + \cdots \]

Sample theorems:

- \(|A \otimes B| = |A| \cdot |B|\), where \(\otimes\) is a certain graph product
- \(|A \cup B| = |A| + |B| - |A \cap B|\), under quite strict hypotheses
- Graph magnitude has other invariance properties shared with the Tutte polynomial.
Magnitude of other enriched categories

Magnitude of \( n \)-categories

- Start with the notion of the size (cardinality) of a finite set.
- Taking \( \mathcal{V} = \text{FinSet} \), automatically get notion of the size (magnitude) of a finite 1-category.
- Taking \( \mathcal{V} = \text{FinCat} \), automatically get notion of the size (magnitude) of a finite 2-category.
- \ldots
- Automatically get notion of the size (magnitude) of a finite \( n \)-category \((n < \infty)\).

Almost nothing is known about this!

And what is the magnitude of an \( \infty \)-category?

Also What about other bases \( \mathcal{V} \) of enrichment?
4. Where’s the category theory?
Overview

magnitude
homology

categories
posets
groupoids

$n$-cats
?
linear cats
graphs
metric spaces
diversity
5. Magnitude homology: a sketch
Two perspectives on Euler characteristic

So far: Euler characteristic has been treated as an analogue of cardinality.

Alternatively: Given any homology theory $H_\ast$ of any kind of object $A$, can define

$$
\chi(A) = \sum_{n=0}^{\infty} (-1)^n \text{rank } H_n(A).
$$

Note:

- $\chi(A)$ is a *number*
- $H_\ast(A)$ is an *algebraic structure*, and functorial in $A$.

In this sense, homology is a categorification of Euler characteristic.
The homology of an ordinary category

Let $\mathbf{A}$ be a small category.

Its nerve $\mathcal{N}\mathbf{A}$ is a simplicial set.

Form the associated chain complex $C_\ast(\mathbf{A})$ in the usual way.

The homology $H_\ast(\mathbf{A})$ of $\mathbf{A}$ is the homology of $C_\ast(\mathbf{A})$.

Theorem $H_\ast(\mathbf{A}) = H_\ast(B\mathbf{A})$.

Hence

$$
\sum_{n=0}^{\infty} (-1)^n \text{rank } H_n(\mathbf{A}) = \sum_{n=0}^{\infty} (-1)^n \text{rank } H_n(B\mathbf{A}) = \chi(B\mathbf{A}) = |\mathbf{A}|.
$$

Goal For a $\mathbb{V}$-category $\mathbf{A}$, define a ‘homology’ $H_\ast(\mathbf{A})$ in such a way that

$$
\sum_{n=0}^{\infty} (-1)^n \text{rank } H_n(\mathbf{A}) = |\mathbf{A}|.
$$

It can be done!
The magnitude homology of a graph

Richard Hepworth and Simon Willerton defined the magnitude homology of a graph $A$.

(Definition omitted here.)

Features:

- It’s a *graded* homology theory, i.e. each $H_n(A)$ is a *graded* abelian group.
- Hence $\chi(A) = \sum(-1)^n \text{rank } H_n(A)$ is a *sequence* of integers.
- Viewing this sequence as a power series over $\mathbb{Z}$, it is exactly the magnitude of $A$.
  So: magnitude homology categorifies magnitude.
- The formulas for $|A \otimes B|$ and $|A \cup B|$ can be categorified to give Künneth and Mayer–Vietoris theorems.
- Magnitude homology can distinguish between graphs that mere magnitude cannot.
The magnitude homology of an enriched category

Let $\mathcal{V}$ be a monoidal category.

Mike Shulman gave a general definition of the magnitude homology $H_*(A)$ of a $\mathcal{V}$-category $A$.

(Definition omitted here.)

Features:

- It generalizes both homology of ordinary categories and magnitude homology of graphs.
- The Euler characteristic of the magnitude homology $H_*(A)$ is the magnitude $|A|$ (in a suitably formal sense).
  So: magnitude homology categorifies magnitude.
- The general definition is a kind of Hochschild homology.
- There’s an accompanying cohomology theory.
The magnitude homology of a metric space

In particular, the general definition gives a homology theory of metric spaces. It’s a genuinely *metric* homology theory — not just topological.

**Sample theorem** For compact $A \subseteq \mathbb{R}^n$,

$$H_1(A) = 0 \iff A \text{ is convex}.$$  

Very recent result of Nina Otter (arXiv paper last Wednesday):  

*magnitude homology is related to (but different from!)*  

*persistent homology.*
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