Families of representations of the Witt algebra

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Definition

The *Witt (or centerless Virasoro) algebra* W is the Lie algebra W with $\mathbb{C}$-basis $\{e_n\}_{n\in\mathbb{Z}}$ and Lie bracket

$$[e_n, e_m] = (m-n)e_{n+m}.$$

Let $U(W)$ be the universal enveloping algebra of W.

$$U(W) = \frac{\mathbb{C}\langle e_n \mid n \in \mathbb{Z}\rangle}{([e_n, e_m] = (m-n)e_{n+m})},$$

which is $\mathbb{Z}$-graded with $\deg e_n = n$.

$U(W)$ is a domain, has infinite global dimension, and has sub-exponential growth.

Theorem (S.–Walton, 2013)

$U(W)$ is neither left nor right noetherian.

Poincaré–Birkhoff–Witt: If L is a finite-dimensional Lie algebra then $U(L)$ is noetherian.

There are no known infinite-dimensional L with $U(L)$ noetherian, and it is conjectured that none exist. Known:

Corollary (S.-Walton, 2013)

If L is infinite-dimensional, $\mathbb{Z}$-graded, simple, and has polynomial growth, then $U(L)$ is not noetherian.

Quick summary of the proof of the theorem.

General fact: if L' is a Lie subalgebra of L and $U(L)$ is noetherian, then $U(L')$ is noetherian.

Definition

The positive (part of the) Witt algebra is defined to be the Lie subalgebra W_+ of W generated by $\{e_n\}_{n\geq 1}$.

We showed that $U(W_+)$ is not left or right noetherian by using geometry to construct a GK-3 homomorphic image of $U(W_+)$.

Then we showed the image is not noetherian.

The construction of $\rho : U(W_+) \to R$.

Notation

Let $X = V(xz - y^2) \subset \mathbb{P}^3$. X is a singular quadric surface: a rational surface whose singular locus is the vertex $[w : x : y : z] = [1 : 0 : 0 : 0]$.

Define $\tau \in \mathrm{Aut}(X)$ by

$$\tau([w : x : y : z]) = [w - 2x + 2z : z : -y - 2z : x + 4y + 4z].$$

τ acts on $\mathbb{C}(X) \cong \mathbb{C}(u, v)$ by pullback; we abuse notation and denote this action by τ as well.

We work in the ring $\mathbb{C}(X)[t; \tau]$, where $tg = g^\tau t$ for all $g \in \mathbb{C}(X)$.

Construct $\rho : U(W_+) \to \mathbb{C}(X)[t;\tau]$.

Proposition

There is a graded algebra homomorphism $\rho : U(W_+) \to \mathbb{C}(X)[t;\tau]$ defined by $\rho(e_1) = t$ and $\rho(e_2) = ft^2$, where

$$f = \frac{w + 12x + 22y + 8z}{12x + 6y} \in \mathbb{C}(X)$$

Proof.

$U(W_+)$ is generated by e_1 and e_2, and has two relations, one in degree 5 and one in degree 7. Check that t and ft^2 satisfy the relations for $U(W_+)$. $\square$

$\rho(U(W_+))$ is a GK-dimension 3 birationally commutative algebra. It[1] is a subalgebra of a twisted homogeneous coordinate ring on X.

- Recall that an algebra is birationally commutative if it is a graded subalgebra of some $K[t;\tau]$, where K is a field.
- Recall also that twisted homogeneous coordinate rings are well-understood algebras with good properties that are built from a projective variety and an automorphism.

$\mathbb{N}$-graded noetherian birationally commutative algebras of GK-dimension 3 are classified. (S.) By using geometry, can analyse $\operatorname{Im}\rho$ and show that it is not noetherian.

[1] More accurately, the 2nd Veronese of $\rho(R)$ is contained in a twisted homogeneous coordinate ring.

Let's write ρ in a nicer way.

Definition

Let

$$S := \mathbb{C}\langle x, y, z\rangle / \left(\begin{array}{c} xy - yx - y^2 \\ xz - zx - yz \\ yz - zy \end{array} \right).$$

S is Artin-Schelter regular. Its graded quotient ring is $\mathbb{C}(\mathbb{P}^2)[t; \tau']$ where

$$\tau' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(In fact S is a twisted homogeneous coordinate ring on $\mathbb{P}^2$ using τ'.)

Proposition

The map $\sigma : U(W_+) \to S$, $e_n \mapsto (x - nz)y^{n-1}$ *induces a ring homomorphism.*

Proof.

$$
\begin{aligned}
&\sigma(e_n e_m - e_m e_n) = \\
&\quad (x - nz)y^{n-1}(x - mz)y^{m-1} - (x - mz)y^{m-1}(x - nz)y^{n-1} \\
&= [x^2 y^{n+m-2} - (n-1)xy^{n+m-1} - (n+m)xy^{n+m-2}z + n^2 y^{n+m-1}z \\
&\qquad + nmy^{n+m-2}z^2] \\
&- [x^2 y^{n+m-2} - (m-1)xy^{n+m-1} - (m+n)xy^{n+m-2}z + m^2 y^{n+m-1}z \\
&\qquad + nmy^{n+m-2}z^2] \\
&= (m-n)xy^{n+m-1} + (n^2 - m^2)y^{n+m-1}z \\
&= \sigma([e_n, e_m]).
\end{aligned}
$$

□

There is a birational map $\phi : \mathbb{P}^2 \dashrightarrow X$ so that

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed, "\phi"] \arrow[d, "\tau'"'] & X \arrow[d, "\tau"] \\ \mathbb{P}^2 \arrow[r, dashed, "\phi"] & X \end{tikzcd}$$

commutes.

So $\mathbb{C}(X)[t; \tau] \cong \mathbb{C}(\mathbb{P}^2)[t, \tau']$, and σ is ρ repackaged.

The height 1 prime ideals of S are:

$$\{(y)\} \cup \{(z - ay) \mid a \in \mathbb{C}\}.$$

(They correspond to τ'-invariant curves on $\mathbb{P}^2$. Note y and $z - ay$ are normal.)

Define:

$$\sigma_\infty : U(W_+) \xrightarrow{\sigma} S \longrightarrow S/(y) \cong \mathbb{C}[x, z]\ .$$

$$\sigma_a : U(W_+) \xrightarrow{\sigma} S \longrightarrow S/(z - ay)\ .$$

This gives a family of factors of $U(W_+)$.

σ_∞ is boring: $\sigma_\infty(e_n) = 0$ for $n \geq 2$, and $\operatorname{Im} \sigma_\infty \cong \mathbb{C}[x]$, $e_1 \mapsto x$.

Fix $a \in \mathbb{C}$.

We have

$$\begin{aligned} \sigma_a : U(W_+) &\to \mathbb{C}\langle x, y\rangle/(xy - yx - y^2) \\ e_n &\mapsto xy^{n-1} - any^n \end{aligned}$$

That is, σ gives a family of maps from $U(W_+)$ to the Jordan plane $\mathbb{C}_J[x, y] = \mathbb{C}\langle x, y\rangle/(xy - yx - y^2)$. We study these maps as a varies.

Lemma

$\mathbb{C}_J[x, y]$ is isomorphic to the <u>twist</u> of $\mathbb{C}[x, y]$ by $\alpha = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$: define

$$f_n \star g_m := fg^{\alpha^n}.$$

Proof.

$$x \star y - y \star x = xy - y(x - y) = y^2 = y \star y$$

□

Note: A twist of a polynomial ring by a graded automorphism is a twisted homogeneous coordinate ring.

So $\sigma_a(e_n) = (x - nay) \star y^{n-1} = (x - nay)y^{n-1}$.

$$\sigma_a(e_i e_{n-i}) = ((x - iay)y^{i-1}) \star (x - (n-i)ay)y^{n-i-1} = \\ (x - iay)(x - [(n-i)a + i]y)y^{n-2}.$$

When $a = 0$, all degree n monomials in the e_i vanish at $[0 : 1]$.

When $a = 1$, all degree n monomials in the e_i vanish at $[n : 1]$.

Otherwise no common vanishing locus.

Writing S as the twist $\mathbb{C}[x, y, z], \star$ of $\mathbb{C}[x, y, z]$ by τ', we obtain that in degree n, the common vanishing locus of $\sigma(U(W_+)_n)$ is

$$\{[0:1:0], [n:1:1] = (\tau')^{-n}([0:1:1])\} \cup \{\text{ complicated vanishing at } [1:0:1] \text{ }\}$$

Since both $[0 : 1 : 0]$ and $[0 : 1 : 1]$ have infinite but not dense orbits, results in the classification of birationally commutative projective surfaces imply that $\sigma(U(W_+))$ is not noetherian. This is basically the proof from our 2013 preprint, except that we worked on X, not $\mathbb{P}^2$.

Historical note: The fact that the subalgebra of S consisting of <u>all</u> functions vanishing at $[0 : 1 : 0]$ is not noetherian is a special case of the main theorem from the talk I gave on idealizers in Shanghai in 2006.

Theorem

When $a = 0$, the image of σ_a is $\mathbb{C} + x\mathbb{C}_J[x, y]$.
When $a = 1$, the image of σ_a is $\mathbb{C} + \mathbb{C}_J[x, y]x$.
For other a, $\sigma_a(U(W_+))$ contains $\mathbb{C}_J[x, y]_{\geq 4}$.

The map σ_0 is the map constructed by Dean and Small in 1990.

Note: The image of σ_a is always noetherian. (Follows from Artin and Stafford's results on noncommutative projective curves, even before we compute the image.)

Non-noetherianness of idealisers is a codimension 2 phenomenon.

Comparing Hilbert series, we see that $\ker \sigma_0$ and $\ker \sigma_1$ contain an element of degree 4.

Theorem

We have $\ker \sigma_0 = \ker \sigma_1$*. As a 2-sided ideal, it is generated by* $e_2 e_1^2 - e_1^2 e_2 + 2e_2^2$.

Computing $\ker \sigma_a$ and $\ker \sigma$ is work in progress.

Let $a, b \in \mathbb{C}$, and consider

$$P_{a,b} := S/S(x - by, z - ay).$$

This is a point module of S: a module with the Hilbert series of $\mathbb{C}[y]$.

The $P_{a,b}$ are all the left S-point modules not killed by y.

In fact, let $S' := S[y^{-1}]$ and let

$$V_{a,b} := S'/S'(x - by, z - ay).$$

The $V_{a,b}$ are the S'-modules with the Hilbert series of $\mathbb{C}[y, y^{-1}]$.

We'll use $v_n := \overline{y^n}$ as the basis of $V_{a,b}$.

The map $e_n \mapsto (x - nz)y^{n-1}$ (now with $n \in \mathbb{Z}$) defines a homomorphism

$$\sigma' : U(W) \to S'.$$

Proposition

$U(W)$ acts on $V_{a,b} := S'/S'(x - by, z - ay)$ by:

$$e_n \cdot v_m = (b - 1 + (1 - a)n + m)v_{n+m}.$$

Proof.

First, z acts as ay on $V_{a,b}$. And we have

$$0 = \overline{y^{n+m-1} \star (x - by)} = x \cdot v_{n+m-1} + (-b - n - m + 1)v_{n+m}.$$

So

$$e_n \cdot v_m = (xy^{n-1} - nay^n) \cdot v_m = (b + n + m - 1 - an)v_{n+m}.$$

$\square$

The representations $V_{a,b}$ are well-known: they are the intermediate series representations of the Witt algebra.

$$V_{a,b} \text{ (Sierra) } = V_{\alpha,\beta} \text{ (Zelmanov)}$$

(where there is a change of basis between a, b and α, β).

We say, imprecisely, that S' is the universal intermediate series representation.

Define

$$\sigma'_a : U(W) \xrightarrow{\sigma'} S' \longrightarrow S'/(z - ay) \cong \mathbb{C}_J[x, y, y^{-1}]$$

All point modules over $\mathbb{C}_J[x, y]$ are faithful and y-torsionfree except for $\mathbb{C}_J[x, y]/(y)$. So we have:

Theorem

The annihilator of $V_{a,b}$ as a $U(W)$-module is $\ker \sigma'_a$ and depends only on a.

This talk is ahistorical: the map ρ that was originally used to show $U(W)$ is not noetherian was built from point modules of $U(W_+)$, rather than constructing point modules from a homomorphic image.

Original goal: study $U(W_+)$ through geometry of representations of W_+. What is the geometry of point modules over W_+?

For this part of the talk, I'll use a different presentation for $U(W_+)$.

Notice:

$$[e_1, [e_1, [e_1, e_2]]] = [e_1, [e_1, e_3]] = 2[e_1, e_4] = 6e_5$$

$$[e_2, [e_2, e_1]] = -[e_2, e_3] = -e_5$$

$$\text{And } [e_1, [e_1, [e_1, [e_1, [e_1, e_2]]]]] = -40[e_2, [e_2, [e_2, e_1]]]$$

Proposition (Ufnarovskii)

Let $x = e_1$, $y = e_2$. Then

$$U(W_+) \cong \frac{\mathbb{C}\langle x, y\rangle}{\begin{pmatrix} [x, [x, [x, y]]] + 6[y, [y, x]], \\ [x, [x, [x, [x, [x, y]]]]] + 40[y, [y, [y, x]]] \end{pmatrix}}.$$

Point modules "look like" W_+. Build up by considering representations that look like $W_+/W_{\geq n+1}$. Let

$$M = \mathbb{C}m_0 \oplus \mathbb{C}m_1 \oplus \cdots \oplus \mathbb{C}m_n$$

be a graded right W_+-module. Also assume that $m_i x = m_{i+1}$ for all $i < n$.

Define $y_i \in \mathbb{C}$ by $m_i y = y_i m_{i+2}$. This gives a map:

$$M \mapsto (y_0, \ldots, y_{n-2}) = \mathbf{y}_M \in \mathbb{A}^{n-1}.$$

Let $V_n = \{\mathbf{y}_M\} \subseteq \mathbb{A}^{n-1}$.

Such M are called <u>truncated point modules</u> and V_n is the n'th <u>point space</u>. (Terminology from noncommutative projective geometry.)

There are maps between the V_n. Consider:

$$\begin{tikzcd}[column sep=small] & (y_0, \ldots, y_{n-2}) \arrow[dl, mapsto, "\phi"'] \arrow[dr, mapsto, "\psi"] & \\ (y_0, \ldots, y_{n-3}) & & (y_1, \ldots, y_{n-2}) \end{tikzcd}$$

We have

$$\phi(\mathbf{y}_M) = \mathbf{y}_{M'} \text{ where } M' = M/M_n.$$

$$\psi(\mathbf{y}_M) = \mathbf{y}_{M''} \text{ where } M'' = (M_{\geq 1})[1].$$

So:

$$\begin{tikzcd} & V_n \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ V_{n-1} & & V_{n-1} \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C}[V_n] & \\ \mathbb{C}[V_{n-1}] \arrow[ur, "\phi"] & & \mathbb{C}[V_{n-1}] \arrow[ul, "\psi"'] \end{tikzcd}$$

Fact: This gives us a ring

$$B = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathbb{C}[V_n] t^n.$$

Multiplication is

$$ft^n g t^m = \phi^m(f) \psi^n(g) t^{m+n} \in \mathbb{C}[V_{n+m}] t^{n+m}.$$

Associativity follows from $\phi\psi = \psi\phi$.

To understand B we need to know the defining equations of the V_n.

Let $\mathbf{y}_M = (y_0, \ldots, y_{n-2}) \in V_n$, where $M = \mathbb{C}m_0 \oplus \cdots \oplus \mathbb{C}m_n$.

We have

$$m_0([x,[x,[x,y]]] + 6[y,[y,x]]) = 0$$

or

$$m_0(x^3y - 3x^2yx + 3xyx^2 - yx^3 + 6y^2x - 12yxy + 6xy^2) = 0$$

or

$$(y_3 - 3y_2 + 3y_1 - y_0 + 6y_0y_2 - 12y_0y_3 + 6y_2y_3)m_5 = 0.$$

Fact: V_n is defined by

$$y_3 - 3y_2 + 3y_1 - y_0 + 6y_0y_2 - 12y_0y_3 + 6y_2y_3 = 0$$

plus equations from $m_i([x,[x,[x,y]]] + 6[y,[y,x]]) = 0$

and $m_i([x,[x,[x,[x,[x,y]]]]] + 40[y,[y,[y,x]]]) = 0$

Go back to multiplication in B. Let

$$X := t \in B_1 = \mathbb{C}[V_1]t \qquad Y := y_0 t^2 \in B_2 = \mathbb{C}[V_2]t^2.$$

Then

$$X^3 Y = t^3 y_0 t^2 = \psi^3(y_0) t^5 = y_3 t^5 \in \mathbb{C}[V_5]t^5.$$
$$X^2 YX = t^2 y_0 t^3 = y_2 t^5 \in \mathbb{C}[V_5]t^5.$$

And

$$X^3 Y - 3X^2 YX + 3XYX^2 - YX^3 + 6Y^2 X - 12YXY + 6XY^2 =$$
$$= (y_3 - 3y_2 + 3y_1 - y_0 + 6y_0 y_2 - 12 y_0 y_3 + 6 y_2 y_3) t^5$$
$$= 0 \text{ in } B_5 = \mathbb{C}[V_5]t^5.$$

Likewise,

$$[X, [X, [X, [X, [X, Y]]]]] + 40[Y, [Y, [Y, X]]] = 0 \text{ in } B_7.$$

From the presentation of $U(W_+)$ we obtain a homomorphism

$$\rho' : U(W_+) \to B$$
$$x \mapsto X, \quad y \mapsto Y.$$

Note: A similar construction gives the map from a Sklyanin algebra to the twisted homogeneous coordinate ring of an elliptic curve, i.e. the "embedding of an elliptic curve in a noncommutative $\mathbb{P}^2$."

To say anything about $\rho'(U(W_+))$ we need some geometry of the V_n.

For $n = 6, 7$, V_n has a component V'_n that is a rational surface, and $\phi, \psi : V'_7 \to V'_6$ are birational.

Let

$$\tau := \psi\phi^{-1} : \mathbb{C}(V'_6) \to \mathbb{C}(V'_6) \cong \mathbb{C}(u, v).$$

This is an automorphism of $\mathbb{C}(u, v)$ and it is our original τ.

By restricting B to the V'_n we construct a map $B \to \mathbb{C}(u, v)[t; \tau]$, and we have:

$$\begin{tikzcd} U(W_+) \arrow[r, "\rho'"] \arrow[rr, bend right, "\rho"'] & B \arrow[r] & \mathbb{C}(u, v)[t; \tau] \end{tikzcd} .$$

Conjecture

A Lie algebra L is finite dimensional if and only if the universal enveloping algebra $U(L)$ is noetherian.

Thank you!