Enveloping algebras of infinite-dimensional Lie algebras

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Outline

1. Lie algebras and enveloping algebras
2. Big enveloping algebras
3. Some conjectures
4. The growth conjecture
5. Progress on the noetherianity conjecture
6. Symmetric powers: a theorem and two more conjectures
Definition

Let $\partial = \frac{d}{dx}$. The Witt algebra is $W = \mathbb{C}[x, x^{-1}]\partial$.

Here $\mathbb{C}[x, x^{-1}] = \text{Laurent polynomials in } x \text{ with complex coefficients, such as } x^2 + 1 \text{ or } x + 3 + x^{-1}$.

$W$ is the algebra of derivations of $\mathbb{C}[x, x^{-1}]$, where a derivation is a function $d : \mathbb{C}[x, x^{-1}] \rightarrow \mathbb{C}[x, x^{-1}]$ that obeys the Leibniz rule:

$$d(fg) = d(f)g + fd(g).$$
In what sense is $\mathcal{W}$ an algebra? Usually: “algebra” means “associative algebra”, i.e., a vector space which is also a (associative) ring.

How can we combine two elements of $\mathcal{W}$ to get a third?

The product rule means that if $f, g \in \mathbb{C}[x, x^{-1}]$, then

$$\partial(gf) = gf' + g'f = (g\partial + g')(f),$$

and so we write

$$\partial g = g\partial + g'.$$

(We can formalise this by talking about “operators on $\mathbb{C}[x, x^{-1}]$”.)
If $\partial g = g\partial + g'$ then

$$f\partial g\partial = fg\partial^2 + fg'\partial \not\in W.$$  

On the other hand,

$$f\partial g\partial - g\partial f\partial = (fg\partial^2 + fg'\partial) - (gf\partial^2 + gf'\partial) = (fg' - gf')\partial \in W.$$  

Define the bracket

$$[f\partial, g\partial] = (fg' - gf')\partial \quad \text{on } W.$$
Properties of $[−, −]$:

- Not associative!
- $\mathbb{C}$-linear in both factors.
- Antisymmetric: $[X, Y] = −[Y, X]$.
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

(1)–(3) define a Lie algebra and so $W$ is a Lie algebra.

Lie algebras are ubiquitous in maths (and physics). Examples:

- $\mathfrak{gl}(n, \mathbb{C}) = \{n \times n$ matrices $\}$, $[A, B] = AB − BA$.
- $\mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C})| \text{tr}(X) = 0\}$
- Any vector space $V$, with $[−, −] = 0$ (abelian)
- If $G$ is a Lie group (manifold which is also a group, like $S^1$), then $T_e G$ is automatically a Lie algebra, possibly over $\mathbb{R}$. The Lie bracket echoes the structure of the group, and the group (near $e$) can be reconstructed from the Lie algebra!
We can turn a Lie algebra into an associative ring.

**Definition**

Let $\mathfrak{g}$ be a Lie algebra. The **universal enveloping algebra** of $\mathfrak{g}$ is

$$U(\mathfrak{g}) = T(\mathfrak{g})/(XY - YX = [X, Y]|X, Y \in \mathfrak{g}).$$

Example: a basis for $\mathfrak{sl}_2 = \mathfrak{sl}(2, \mathbb{C})$ is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $U(\mathfrak{sl}_2)$ consists of noncommutative polynomials in $e, f, h$, subject to the rules:

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f.$$
Fact (Poincaré-Birkhoff-Witt theorem): monomials $e^i f^j h^k$ give a basis for $U(\mathfrak{sl}_2)$.

Likewise, in $W = \mathbb{C}[x, x^{-1}] \partial$ let $e_n = x^{n+1} \partial$, so in $U(W)$

$$e_n e_m - e_m e_n = [e_n, e_m] = \left(x^{n+1}(x^{m+1})' - x^{m+1}(x^{n+1})'\right) \partial$$

$$= (m - n)e_{n+m}.$$

Then monomials $e^{i_1}_{n_1} e^{i_2}_{n_2} \ldots e^{i_k}_{n_k}$ with $n_1 < n_2 < \cdots < n_k$ form a basis for $U(W)$. 
Fact: If \( \dim \mathfrak{g} = d < \infty \), then \( U(\mathfrak{g}) \) has all the nice properties of \( \mathbb{C}[x_1, \ldots, x_d] \), but is more interesting.

- \( U(\mathfrak{g}) \) is \((L + R)\) noetherian: L + R ideals are finitely generated.
- \( U(\mathfrak{g}) \) has polynomial growth: if \( V \subset U(\mathfrak{g}) \) is fin. dim. with \( 1 \in V \), then \( \dim V^n \sim n^d \)
- 2-sided ideals of \( U(\mathfrak{g}) \) are much harder to understand than those of \( \mathbb{C}[x_1, \ldots, x_d] \)
Prime ideals of $\mathbb{C}[x_1, x_2, x_3]$ correspond to subvarieties of $\mathbb{C}^3$:

- $\mathbb{C}^3$
- surfaces in $\mathbb{C}^3$
- curves
- points

Picture due to Brent Pym
Prime ideals of $U(\mathfrak{sl}_2)$ are

- $(0)$
- $I_\lambda$ for $\lambda \in \mathbb{C}$
- $J_n$ for $n \in \mathbb{Z}$

\[
\begin{array}{cccccc}
& & & & &  J_n \\
& & & & I_\lambda & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(0) & & & & & \\
\end{array}
\]
Enveloping algebras of finite-dimensional Lie algebras are
- fundamental examples of well-behaved noncommutative rings
- very well studied
- deep links with geometry
**Definition**

A big enveloping algebra is $U(\mathfrak{g})$ where $\dim \mathfrak{g} = \infty$.

Fundamental question: Are big enveloping algebras ever nice?

**Theorem (M. Smith, 1976)**

$\dim \mathfrak{g} < \infty \iff U(\mathfrak{g})$ has polynomial growth.

Example: if $1 \in V \subset U(W)$, then $\dim V^n \sim e^{\sqrt{n}}$ (“subexponential growth”)

**Question (Amayo-Stewart 1974)**

*If $\dim \mathfrak{g} = \infty$, can $U(\mathfrak{g})$ ever be noetherian?*
Note: if \( g \) is abelian then \( U(g) = \mathbb{C}[g] \) and is noetherian \( \iff \dim g < \infty \).

**Question (Dean-Small 1990)**

*Is \( U(W) \) noetherian?*

**Theorem**

*(S.–Walton 2013) No. \( U(W) \) is neither left nor right noetherian.*
Outline of proof:

Consider the ring $A = \mathbb{C}\langle x, x^{-1}, \partial \rangle$. Here $\partial$ is still $\frac{d}{dx}$ and we have $\partial x = x\partial + 1$ (another consequence of the product rule).

There is a ring homomorphism $\phi : U(W) \to A$ given by $f\partial \mapsto f\partial$ – that is, get $\phi$ from $W \subset A$.

It turns out that $\ker \phi$ is:

- not finitely generated as either a left or a right ideal
- principal as a two-sided ideal (Conley-Martin 2006).
Corollary (S.-Walton 2013)

\( U(\mathfrak{g}) \) is not left or right noetherian for \( \mathfrak{g} = \)

- simple, \( \mathbb{Z} \)-graded, polynomial growth
- the Virasoro algebra

Also not left or right noetherian for \( \mathfrak{g} = \)

- a generalized Witt algebra (S. - Špenko, 2016)
- \( \infty \)-dimensional filiform
- \( \infty \)-dimensional Kac-Moody
- \ldots

In fact there is no known example of an infinite-dimension Lie algebra with a left or right noetherian enveloping algebra.
Conjecture

(Dixmier? S.-Walton 2013) Let $\mathfrak{g}$ be a Lie algebra. Then $U(\mathfrak{g})$ is left and right noetherian if and only if $\mathfrak{g}$ is finite-dimensional.

This is hard! Let’s look at $W$ and make some conjectures about $U(W)$. 
Let $B = \phi(U(W)) \subset A = \mathbb{C}\langle x, x^{-1}, \partial \rangle$. Some facts about $B$:

- $B$ satisfies the ascending chain condition on 2-sided ideals

ACC: if $I_1 \subseteq I_2 \subseteq \ldots$ are ideals, then $\exists n : I_n = I_{n+1} = \ldots$ (equivalently: 2-sided ideals are finitely generated)

- Thus $U(W)$ has ACC on 2-sided ideals containing $\ker \phi$.
- Remember: $\ker \phi$ is principal as a 2-sided ideal.
- $B$ has polynomial growth, whereas $U(W)$ does not.

This suggests that “ideals in $U(W)$ are big (and sparse)”
Conjecture (Growth conjecture, Petukhov-S. 2017)

*Any proper factor of $U(W)$ has polynomial growth (“Ideals are big”)*

Suggested by computer experiments of I. Stanciu in 2016-17.

If ideals are big, then there probably aren’t very many of them.

Conjecture (Noetherianity conjecture, Petukhov-S. 2017)

*$U(W)$ satisfies the ascending chain condition on 2-sided ideals. (“Ideals are sparse”)
Theorem (Growth conjecture theorem, Iyudu-S., 2018)

The growth conjecture holds: any proper factor of $U(W)$ has polynomial growth.

The proof uses the Poisson algebra structure on the symmetric algebra $S(W) = \mathbb{C}[\ldots, e_{-1}, e_0, e_1, e_2, \ldots]$ (commutative).

There’s a surjective linear map $\text{gr} : U(W) \to S(W)$, defined by:

$$\text{gr}(e_{n_1}^{i_1} e_{n_2}^{i_2} \ldots e_{n_k}^{i_k}) = e_{n_1}^{i_1} e_{n_2}^{i_2} \ldots e_{n_k}^{i_k}.$$

Define a (well-defined!) bracket $\{−, −\}$ on $S(W)$ by:

$$\{\text{gr}(A), \text{gr}(B)\} = \text{gr}(AB − BA),$$

so $\{e_n, e_m\} = (m − n)e_{n+m}$. 
The bracket $\{−, −\}$ gives $S(W)$ a Poisson algebra structure.

It turns out that if $I \triangleleft U(W)$, then $\text{gr}(I)$ is not only an ideal of $S(W)$, but a Poisson ideal: given $A \in \text{gr}(I)$, then $\{B, A\} \in \text{gr}(I)$ for all $B \in S(W)$.

The proof studies Poisson ideals of $S(W)$. Main idea: if we think of elements of $U(W)$ as “noncommutative polynomials”, then elements of $S(W)$ are their “leading terms”. The Poisson ideals of $S(W)$ capture just enough information about ideals of $U(W)$, and are easier to analyse because we’re only looking at one term.
Look at $\mathcal{W}_+ = \mathbb{C}(e_n : n \geq 1) = x^2 \mathbb{C}[x] \partial$ (still a Lie algebra).

Again, we work with the Poisson structure on $S(\mathcal{W}_+)$. 

**Proposition (Petukhov-S., 2017)**

$S(\mathcal{W}_+)$ has ACC on radical Poisson ideals.

(Recall: $I$ is **radical** if $f^n \in I \Rightarrow f \in I$.)

Follows from:

**Lemma**

*Let $I$ be a radical Poisson ideal of $S(\mathcal{W}_+)$. Then $S(\mathcal{W}_+)/I$ embeds in a finitely generated commutative algebra. (Which has ACC on all ideals, by the Hilbert Basis Theorem.)*
Outline of proof of lemma:

Let $f \in I$ of minimal degree, let $e_n$ be biggest variable in $f$.

\[ \{e_1, f\} = e_{n+1}p + q \in I, \quad \text{where } p, q \in \mathbb{C}[e_1, \ldots, e_n]. \]

Then $\deg p < \deg f$ so $p \not\in I$.

Now $p \not\in I$ and so $I = (I : p) \cap J$ where $p \in J$, and $J$ is radical and Poisson.

Here $(I : p) = \{g | pg \in I\}$.

By induction $S(W_+)/J \hookrightarrow$ a finitely generated algebra $C$. 
Now consider $S(W_+)/(I : p) \hookrightarrow (S(W_+)/I)[p^{-1}]$.

In $(S(W_+)/I)[p^{-1}]$ we have $e_{n+1} = -qp^{-1}$,
$e_{n+2} = (\text{something})p^{-2}$, etc.

So $(S(W_+)/I)[p^{-1}]$ is finitely generated and

$$S(W_+)/I \hookrightarrow S(W_+)/(I : p) \oplus S(W_+)/J \hookrightarrow (S(W_+)/I)[p^{-1}] \oplus C.$$

Then push ACC on (radical) ideals of a finitely generated commutative algebra down to $S(W_+)/I$ and thus to ACC on radical ideals of $S(W_+)$. 
Now since if $J \triangleleft U(W_+)$ then $\text{gr} J$ is a Poisson ideal of $S(W_+)$, we deduce:

**Corollary (Petukhov-S.)**

$U(W_+) \text{ has ACC on ideals } I \text{ such that } \text{gr}(I) \text{ is radical.}$
Define $S^n(W_+)$ to be the vector subspace of $S(W_+)$ spanned by monomials of degree $n$ in the $e_i$. (So $S^1(W_+) = W_+$.)

Let $m \in S^n(W_+)$ and define $e_i \cdot m = \{e_i, m\}$, which is still in $S^n(W_+)$. This gives $S^n(W_+)$ the structure of a $U(W_+)$-module (= representation of $W_+$).

**Theorem (Petukhov-S., 2017)**

$S^2(W_+)$ is a noetherian representation of $W_+$. Thus $U(W_+)$ has ACC on ideals generated by quadratic expressions in the $e_i$.

The proof of the theorem uses (lots of) explicit computer calculations. In other words, it’s not pretty.
I close with two final conjectures.

**Conjecture (Symmetric power conjecture)**

For $n \in \mathbb{N}$, $S^n(W_+)$ and $S^n(W)$ are noetherian.

Further, there is an attractive proof of this fact.

**Conjecture**

The symmetric power conjecture implies the noetherianity conjecture (for $W_+$ and $W$).